



When Leisure Becomes Excessive: A Bifurcation Result in Endogenous Growth Theory

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resumo

résumé / abstract

O pressuposto tradicional no que respeita à oferta endógena de trabalho em modelos de crescimento económico é o de que a utilidade aumenta com o lazer, independentemente da afectação específica de tempo do agente representativo num dado momento. Nesta nota explora-se as consequências, sobre a estabilidade dinâmica, de assumir que o agente prefere não ter tempo livre em excesso, isto é, de considerar que a utilidade marginal do lazer não é necessariamente positiva para qualquer valor da sua percentagem de tempo de lazer (em particular, para valores elevados desta percentagem). Ao incluir este pressuposto num modelo de crescimento endógeno AK típico, revela-se que o sistema tenderá a permanecer, independentemente dos valores de parâmetros, sobre uma linha de bifurcação.

La présupposition traditionnelle en ce qui concerne l'offre endogène de travail dans des modèles de croissance économique est que l'utilité augmente avec le loisir et ce, indépendamment de l'affectation spécifique de temps de l'agent représentatif à un moment donné. Dans cet article, nous exploitons les conséquences, sur la stabilité dynamique, du fait d'assumer que l'agent préfère ne pas avoir trop de temps libre, c'est-à-dire, de considérer que l'utilité marginale du loisir n'est pas nécessairement positive pour une quelconque valeur de son pourcentage de temps de loisir (notamment, pour des valeurs élevées de ce pourcentage). En incluant cette présupposition dans un modèle de croissance endogène AK typique, nous constatons que le système tendra à rester, indépendamment des valeurs de paramètres, sur une ligne de bifurcation.

The traditional assumption concerning endogenous labor supply in models of economic growth is that utility increases with leisure, independently of the specific time allocation of the representative agent observed at a given moment. In this note, we explore the consequences, over dynamic stability, of assuming that the agent dislikes having free time in excess, i.e., of considering that the marginal utility of leisure is not necessarily positive for every value of the leisure share (in particular, for high values of this share). By including this assumption in a typical AK endogenous growth model, we find that the system will rest, independently of parameter values, on a bifurcation line.

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1. Introduction

In modern societies it is legitimate to ask whether there is always a positive relation between leisure and the utility leisure brings. Unemployed people or people working at partial time often dislike having too much free time and most of us find personal realization and social recognition in work. This idea is not taken into account by macroeconomic models, namely the models of growth and business cycles in which the endogenous determination of the labor-leisure choice is central to the analysis (among many others, these models include Stokey and Rebelo, 1995; Ortigueira, 2000; and Duranton, 2001).

In this note, we explore the consequences of assuming that the representative agent prefers an intermediate share of leisure time than extreme values (no leisure or too much leisure). These consequences are addressed under an endogenous growth setup. Interesting dynamic results are obtained in what concerns, both, transitional dynamics and the long term balanced growth path.

The argument we propose is not uncontroversial and it should be understood in relative terms. Some societies value leisure more than others and, thus, the point in which individuals begin to withdraw less utility from leisure as leisure rises varies from one social context to another. For instance, Glaeser, Sacerdote and Scheinkman (2003) and Blanchard (2004) highlight the difference in hours worked in Europe and North-America; the difference can be explained, in the view of these authors, by a cultural predilection for leisure that has gained weight in Europe since the 1960s. Americans work more hours not only because this allows them to increase their income but also because there is a cultural context that inhibits individuals from getting utility from leisure when they do not participate or participate scarcely on the effort of creating value to the economy they belong to. Although this seems a more pronounced tendency in North-America than in Europe, this is indeed a trend that we can identify all over the developed world.

The note is organized as follows. Section 2 describes the model's features, section 3 addresses the properties of the steady-state, section 4 analyzes local stability, and section 5 concludes.

2. A Model of Excess of Leisure

Assume a representative agent that maximizes the following sequence of utility functions over an infinite horizon,

$$U_0 = \sum_{t=0}^{+\infty} \beta^t \cdot U(c_t, L_t) \quad (1)$$

In expression (1), c_t stands for the real level of consumption and $L_t \in (0,1]$ is the share of the agent's time allocated to leisure; we let $L_t \equiv 1-l_t$, and hence we designate the share of time allocated to work by l_t . Parameter $0 < \beta < 1$ is the discount factor. The utility function takes the following functional form,

$$U(c_t, L_t) = \ln \left(\frac{c_t}{L_t^{m \cdot L_t}} \right), \quad m > 0 \quad (2)$$

Under function (2), consumption and leisure produce utility separately. Concerning consumption, marginal utility is positive and diminishing, as conventionally assumed. In what respects leisure, we consider that utility rises with leisure when the time allocated to working hours is relatively high, but as the unoccupied time increases the utility withdrawn from leisure falls – the representative agent dislikes having too much free time. Parameter m measures the weight of leisure utility in the overall utility function. In the extreme case in which $m = 0$, function (2) reduces to $U(c_t) = \ln(c_t)$, i.e., the representative agent withdraws utility only from consumption. The larger the value of m , the more the agent attributes relevance to the role of available time in

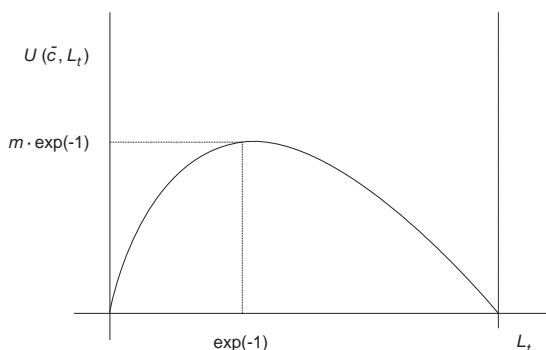


terms of the accomplished level of utility. Thus, utility function (2) can be interpreted as follows: it is an additively separable function with respect to its two arguments and the only unconventional assumption is that positive marginal utility of leisure lasts only until some threshold value; after that, additional leisure leads to decreasing utility.

While we will work, for analytical convenience, with the functional form in (2), it is useful to admit that this function belongs to the following generic class of utility functions: $U: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$, where the following partial derivatives hold, (i) $U_c > 0$; (ii) $U_{cc} < 0$; (iii) $U_L > 0$ if $U(\bar{c}, L_t) < \zeta$, $U_L < 0$ if $U(\bar{c}, L_t) > \zeta$, (with ζ a positive value below unity); (iv) $U_{LL} < 0$; finally, (v) $U_{cL} = U_{Lc} = 0$.

Figure 1 represents the relation between leisure and utility for a constant level of consumption $\bar{c} = 1$. In the limit circumstance in which the agent does not work at all, she will not withdraw any utility from the free time she possesses.

Figure 1 – The utility of leisure



The resource constraint is a trivial capital accumulation equation,

$$k_{t+1} = \tilde{y}_t - c_t + (1 - \delta) \cdot k_t, k_0 \text{ given} \tag{3}$$

Variable k_t respects to the stock of physical capital, $0 < \delta < 1$ is the rate of capital depreciation and \tilde{y}_t stands for the effective level of income. This last variable is considered in contrast with the potential level of income, which is given by a constant returns production function, i.e., $y_t = Ak_t$, with $A > 0$ the level of technology. The potential level of income is defined as the output that is generated when the available working hours are integrally used in production. Normalizing the amount of the agent's time to 1, the amount of effective working hours is l_t . Therefore, assuming that production is proportional to the number of hours worked, we should consider $\tilde{y}_t = Al_t \cdot k_t$. As with the consumption function, the production function is just a particular form of a more general specification one can account for: $\tilde{y}_t = f(l_t, k_t)$, with $f: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and (i) $f_l > 0$, (ii) $f_{ll} = 0$, (iii) $f_k > 0$, (iv) $f_{kk} = 0$, (v) $f_{kl} = f_{lk}$ (equal to a positive constant).

We choose to develop the model in discrete time, but the same type of theoretical structure could be implemented in continuous time. One should not expect to find significant differences relatively to the results we will obtain in the following sections if, instead of taking the maximization of (1) subject to (3), we took the maximization of $U_0 = \int_0^{+\infty} \exp(-\rho \cdot t) \cdot U[c(t), L(t)] \cdot dt$ subject to $\dot{k}(t) = \tilde{y}(t) - c(t) - \delta \cdot k(t)$. In this problem, $c(t)$, $L(t)$, $l(t)$, $\tilde{y}(t)$ and $k(t)$ are the



continuous time versions of the presented variables; $\rho > 0$ is the rate of time preference [the counterpart of this parameter in discrete time is $\rho = (1 - \beta)/\beta$]; $\dot{k}(t) = dk(t)/dt$ refers to the time variation of the capital stock; effective output is $\tilde{y}(t) = f[l(t), k(t)]$.

3. Steady-State Existence and Uniqueness

Let p_t be the co-state variable of k_t . The current-value Hamiltonian function of the proposed problem is:

$$\mathcal{H}(k_t, p_t, c_t, L_t) = U(c_t, L_t) + \beta \cdot p_{t+1} \cdot (\tilde{y}_t - c_t - \delta \cdot k_t)$$

First-order conditions are,

$$\mathcal{H}_c = 0 \Rightarrow \beta \cdot p_{t+1} = c_t^{-1}$$

$$\mathcal{H}_l = 0 \Rightarrow m \cdot [1 + \ln(L_t)] + \beta \cdot p_{t+1} \cdot A \cdot k_t = 0$$

$$\beta \cdot p_{t+1} - p_t = -\mathcal{H}_k \Rightarrow p_t = (1 + A \cdot l_t - \delta) \cdot \beta \cdot p_{t+1}$$

$$\lim_{t \rightarrow +\infty} k_t \cdot \beta^t \cdot p_t = 0 \quad (\text{transversality condition})$$

Under an endogenous growth setup, we define the steady state as the long run locus in which: i) the labor share is constant, $l^* \equiv l_{t+1} = l_t$, and ii) consumption and capital grow at a same rate,

$\frac{c_t}{k_t} \equiv \frac{c_{t+1}}{k_{t+1}} = \frac{c_t}{k_t}$; the second point is a straightforward consequence of the first, given the shape of constraint (3). This is the standard way in which the steady state is defined in endogenous growth models. See in this respect, for instance, McGrattan (1998), Milesi-Ferretti and Roubini (1998) or Gómez (2003: 411); this last one clearly states that 'Along the balanced growth path, consumption, investment in physical and human capital, and the stocks of physical and human capital grow at the same constant rate g , and factor allocations remain constant'.

Proposition 1. *The steady state exists and it is unique.*

Proof: Let $\gamma > -1$ be the growth rate of k_t and c_t in the steady state. Hence, we can define variables that do not grow in the balanced growth path, $\hat{k}_t \equiv \frac{k_t}{(1+\gamma)^t}$ and $\hat{c}_t \equiv \frac{c_t}{(1+\gamma)^t}$. From the optimality conditions, after replacing the original variables by \hat{k}_t and \hat{c}_t , we get the following system,

$$\hat{k}_{t+1} = \frac{1 + A \cdot l_t - \delta}{1 + \gamma} \cdot \hat{k}_t - \frac{1}{1 + \gamma} \cdot \hat{c}_t \quad (4)$$

$$\hat{c}_{t+1} = \frac{\beta}{1 + \gamma} \cdot (1 + A \cdot l_{t+1} - \delta) \cdot \hat{c}_t \quad (5)$$

with



$$l_t = 1 - 1 / \exp\left(\frac{A}{m} \cdot \frac{\hat{K}_t}{\hat{C}_t} + 1\right)$$

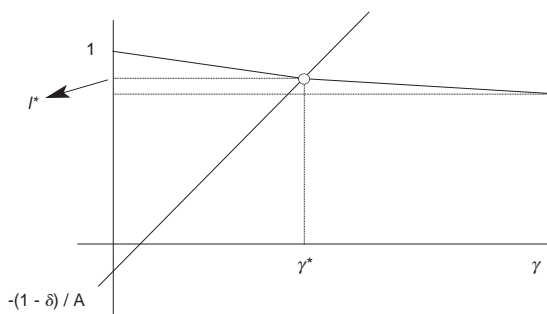
Computing steady state relations, one obtains

$$\frac{\hat{C}^*}{\hat{K}^*} = \frac{1 - \beta}{\beta} \cdot (1 + \gamma) \tag{6}$$

$$l^* = \frac{1 + \gamma - \beta \cdot (1 - \delta)}{\beta \cdot A} = 1 - 1 / \exp\left(\frac{A}{m} \cdot \frac{\beta}{1 - \beta} \cdot \frac{1}{1 + \gamma} + 1\right) \tag{7}$$

The steady-state exists in the form we have defined it (a constant consumption-capital ratio and a constant labor share). To confirm that the steady state is unique, one just has to prove that the growth rate γ is, under (7), a unique value. We have two expressions involving γ . The first one is a linear function with a positive slope, that starts at $-(1 - \delta) / A$ (this is the value for which $\gamma = -1$); note that the constraint $\beta \cdot (1 - \delta) \cdot 1 + \gamma \leq \beta \cdot (1 + A - \delta)$ must hold in order for l^* to be an admissible value. The second expression corresponds to a decreasing function of γ , with l^* tending to 1 when γ tends to -1 and with l^* converging to $1 - 1/\exp(1) \approx 0.6321$ when γ tends to infinity. These two lines intersect in one and only one point and, thus, the steady state growth rate is unique; also unique is the share of time allocated to labor and, according to (6), the consumption-capital share. Figure 2 draws the intersection between the two expressions in (7).

Figure 2 – Growth rate uniqueness



4. Local Dynamics

We are concerned with understanding if the steady state as defined in the previous section is achievable independently of initial values (k_0, c_0) in the vicinity of the steady state, i.e., if there is local stability. The evaluation of the dynamics in the neighbourhood (k^*, c^*) produces a bifurcation result, as stated in proposition 2.

Proposition 2. *The system rests on a bifurcation line, i.e., one of the eigenvalues of the Jacobian matrix of the system is equal to 1. The other eigenvalue locates outside the unit circle.*

Proof: Linearizing the system in the steady state vicinity, one obtains:

$$\begin{bmatrix} \hat{k}_{t+1} - \hat{k}^* \\ \hat{c}_{t+1} - \hat{c}^* \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} + \sigma & -\frac{1}{\alpha} \cdot \left(\frac{1-\beta}{\beta} + \sigma \right) \\ \alpha \cdot \sigma & 1 - \sigma \end{bmatrix} \cdot \begin{bmatrix} \hat{k}_t - \hat{k}^* \\ \hat{c}_t - \hat{c}^* \end{bmatrix} \quad (8)$$

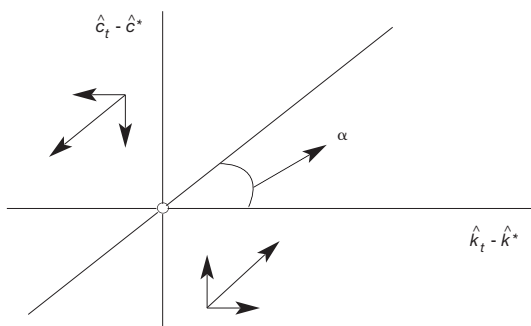
with $\alpha \equiv \frac{1-\beta}{\beta} \cdot (1 + \gamma) > 0$ and $\sigma \equiv \frac{A \cdot [(1-\beta) \cdot (1+A-\delta) - \alpha]}{m \cdot \alpha^2 \cdot \beta} > 0$. The value of σ can be more elegantly presented as $\sigma \equiv \frac{1-\beta}{\beta} \cdot \left(\frac{A}{\alpha} \right)^2 \cdot \frac{L^*}{m}$, if one takes in consideration the steady state value of leisure $L^* = 1 - I^* = \frac{\beta \cdot (1+A-\delta) - (1+\gamma)}{\beta \cdot A}$.

System (8) is derived in appendix.

The trace and the determinant of the Jacobian matrix in (8) are $Tr(J) = (1 + \beta) / \beta$ and $Det(J) = 1 / \beta$. Thus, the system rests over the bifurcation line $1 - Tr(J) + Det(J) = 0$ and the eigenvalues of the Jacobian matrix are $\lambda_1 = 1$ and $\lambda_2 = Det(J) = 1 / \beta > 1$.

Figure 3 sketches the phase diagram of this system. The represented line corresponds to the isoclines of system (8) (i.e., $\hat{k}_{t+1} - \hat{k}_t = 0$ and $\hat{c}_{t+1} - \hat{c}_t = 0$), which are, in the present case, coincidental and equal to $\hat{c}_t - \hat{c}^* = \alpha \cdot (\hat{k}_t - \hat{k}^*)$. As one observes, the steady state will not be reached, unless the initial point is already the steady-state or if any disturbance on the value of consumption is provoked by the representative agent.

Figure 3 – Phase diagram



Because consumption is a control variable, one might expect the representative agent to choose a level of consumption capable of placing the system on the steady state given by (6) and (7). A relevant result of the model relates to the location of the steady state value of leisure; it is important to clarify in which circumstances this long term optimal value corresponds to a point in which the marginal utility of leisure is positive or a point such that the referred marginal utility is negative. The first case occurs for $L^* < \exp(-1) \Leftrightarrow \frac{A}{\alpha \cdot m} > 0$; since A , m and α (the steady state consumption-capital ratio) are positive, the condition holds. Therefore, a central conclusion of the proposed setup is that in the long term the economy is located in a point in which an increase in leisure time implies an increase in utility. We cannot have a steady state leisure share above 36.79% [i.e., above $\exp(-1)$] as a result of solving the agent's maximization problem. In terms of local dynamics, in the vicinity of the steady state, the proposed analysis is not different



from an analysis of a conventional utility function with leisure, since we are working in the increasing segment of the utility function (when considering separately each of its arguments).

5. Discussion

We have developed an endogenous growth model with endogenous labor supply. Differently from the conventional assumption that utility increases with leisure independently of its amount, we have assumed that leisure in excess is less valued by the representative agent than a relatively intermediate level of leisure. The individual does not withdraw too much utility from too much spare time. In the limit, if the agent does not work at all, no utility comes from leisure, exactly as if all the available time was allocated to work.

The imposed assumption allows to find a unique steady state characterized by the existence of a unique growth rate, which cannot be presented explicitly, and a consumption-capital ratio, that is as much higher as the higher is the economy's growth rate; thus, the balanced growth path evidences the idea that the more the economy grows, the more the representative agent is able to consume per unit of physical capital. Another steady state result concerns the labor share: the faster is the pace of growth of the economy, the less the representative agent allocates time to working hours; this can be confirmed by looking at Figure 2.

The central result is that the unconventional form of the utility function concerning leisure produces a bifurcation-instability outcome. Through the linearization of the system around the steady state, we compute a dimension 2 Jacobian matrix with an eigenvalue equal to 1 and the other eigenvalue higher than 1. Any form of stability (a stable node, a stable focus or a saddle-path stable equilibrium) is ruled out. In fact, this result is the corollary of realizing that the optimal steady state value of the leisure share has to be located in the increasing sector of the leisure utility function (i.e., when positive marginal utility from leisure is evidenced). Therefore, the local analysis will focus on the part of the utility function where this has the standard increasing and concave shape and, as a result, a unique stability outcome is accomplished.

The results should be compared with the ones of a similar model with conventional positive marginal utility of leisure. With an utility function $U(c_t, L_t) = \ln c_t + n \cdot L_t$, $n > 0$, solving the same maximization problem, one obtains a constant over time share of labor, $l_t = \frac{A/n - (1 - \beta) \cdot (1 - \delta)}{(1 - \beta) \cdot A}$

and a constant consumption-capital ratio $\frac{c_t}{k_t} = \frac{A}{n}$; consumption and capital grow at rate $g = \frac{\beta}{1 - \beta} \cdot \frac{A}{n} - 1$. Therefore, basically, the assumption of leisure in excess introduces

transitional dynamics over an endogenous growth model that under a trivial framework can be described as being permanently on a balanced growth path. Furthermore, we understand that locally the proposed model does not depart from the expected outcome in a trivial growth analysis, but it introduces a relevant constraint: the long term optimal result excludes a relative amount of working hours (below a given threshold value) that induces a decreasing utility from leisure.

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**Appendix – Derivation of the Linearized System**

The linearization of equation (4), in the neighbourhood of (K^*, c^*) , yields,

$$\hat{k}_{t+1} - \hat{k}^* = \left(\frac{1}{\beta} + \sigma\right) \cdot (\hat{k}_t - \hat{k}^*) - \frac{1}{\alpha} \cdot \left(\frac{1 - \beta}{\beta} + \sigma\right) \cdot (\hat{c}_t - \hat{c}^*) \quad (\text{a1})$$

Relatively to equation (5), this can be rewritten as,

$$\hat{c}_t = \frac{\alpha}{1 - \beta} \cdot \frac{1}{1 + A \cdot l_{t+1} - \delta} \cdot \hat{c}_{t+1} \quad (\text{a2})$$

The linearization of (a2) around (K^*, c^*) allows to write

$$\hat{c}_t - \hat{c}^* = -\alpha \cdot \beta \cdot \sigma \cdot (\hat{k}_{t+1} - \hat{k}^*) + (1 + \beta \cdot \sigma) \cdot (\hat{c}_{t+1} - \hat{c}^*) \quad (\text{a3})$$

Having (a1) in consideration, we rearrange (a3) to present it as follows,

$$\hat{c}_{t+1} - \hat{c}^* = (\alpha \cdot \sigma) \cdot (\hat{k}_t - \hat{k}^*) + (1 - \sigma) \cdot (\hat{c}_t - \hat{c}^*) \quad (\text{a4})$$

Equations (a1) and (a4) form system (8).