The Mixed Mathematical Intermediates

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ABSTRACT

In *Metaphysics* B.2 and M.2, Aristotle gives a series of arguments against Platonic mathematical objects. On the view he targets, mathematicals are substances somehow intermediate between Platonic forms and sensible substances. I consider two closely related passages in B2 and M.2 in which he argues that Platonists will need intermediates not only for geometry and arithmetic, but also for the so-called mixed mathematical sciences (mechanics, harmonics, optics, and astronomy), and ultimately for all sciences of sensibles. While this has been dismissed as mere polemics, I show that the argument is given in earnest, as Aristotle is committed to its key premises. Further, the argument reveals that Annas' uniqueness problem (1975, 151) is not the only reason a Platonic ontology needs intermediates (according to Aristotle). Finally, since Aristotle's objection to intermediates for the mixed mathematical sciences is one he takes seriously, so that it is unlikely that his own account of mathematical objects would fall prey to it, the argument casts doubt on a common interpretation of his philosophy of mathematics.

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1. INTRODUCTION

Much of the literature on Aristotle's objections to Platonic mathematical objects is concerned with assessing the accuracy of Aristotle's reports. My focus is rather on Aristotle's own reasoning about these objects. Besides some discussion of what Annas 1975 names the "uniqueness problem", little has been said about this.[1] I examine two closely related passages (*Metaphysics* B.2 997b12–24 and M.2 1076b39-1077a9), where Aristotle argues that Platonists[2] will need intermediates not only for the pure mathematical sciences, but also for the mixed mathematical sciences, and ultimately for all sciences of sensibles. This is generally seen as mere polemics, and so of little interest: Aristotle is — rather unfairly — piling on absurdities in order to score points against his opponents. My aim is to show that the argument reveals another reason a Platonic ontology needs intermediates (at least according to Aristotle), and that this is in fact a serious argument for him, as he is himself committed to its key premises. Consequently, a careful examination of the argument sheds light on Aristotle's own view about mathematical objects, which should avoid the objections he raises against his opponents.

2. AN ACCUMULATION OF INTERMEDIATES IN *METAPHYSICS* B.2 AND M.2

Aristotle and his Platonic opponents agree that mathematical propositions are not true of sensible things as such.[3] But neither, it seems, can they be true of Platonic forms. Annas (1975) argues that this is due to the uniqueness problem (151). If each form is unique, then there will be only one form of two, one form of three, and so on. Hence mathematical statements such as 2+2=4 cannot be true of forms.[4] There must therefore be non-sensible substances of which mathematical statements can be true, and there must be many of the same kind — e.g. many twos, many triangles, etc. Aristotle reports that there are many (or even an unlimited number) of each kind of intermediate (*Metaphysics* A.6 987b14-18, B.6 1002b14-16, 21-22). Although he does not state that the uniqueness problem is what motivates Platonists to posit intermediates, Annas argues persuasively that this is strongly implied. She adds that this is "the sole line of argument suggested by Aristotle's references to the intermediates" (151; see also 152), and that on his view the intermediates are posited solely as a solution to the uniqueness problem, which is a problem only for the mathematical sciences (156).

This may well be true for arithmetical and geometrical intermediates. However, for the other mathematical sciences — the ones Aristotle calls "the more natural branches of mathematics" (*Physics* 2.2 194a7–8) — there is an additional reason why Platonists ought to supplement their ontology with intermediates. Like the uniqueness problem, this one is unstated but strongly implied. The B.2 and M.2 passages suggest that Aristotle sees intermediates for the mixed sciences as entities his opponents should accept, given their commitment to arithmetical and geometrical intermediates. The arguments run as follows:

> B.2 997b12-24: Further, if we are to posit besides the forms and the sensibles the intermediates between them, we shall have many difficulties. For clearly on the same principle there will be lines besides the lines-in-themselves and the sensible lines, and so with each of the other classes of things; so that since as-

tronomy is one of these mathematical sciences there will also be a heaven besides the sensible heaven, and a sun and a moon (and so with the other heavenly bodies) besides the sensible ones. Yet how are we to believe these things? It is not reasonable even to suppose these bodies immovable, but to suppose their *moving* is quite impossible. And similarly with the things of which optics and mathematical harmonics treat. For these also cannot exist apart from the sensible things, for the same reasons. For if there are sensible things and sensations intermediate between form and individual, evidently there will also be animals intermediate between animals-in-themselves and the perishable animals.[5]

M.2 1076b39-1077a9: Moreover, how can we solve the difficulties reviewed in the *Discussion of Problems*? There will be objects of astronomy over and above perceptible objects, just like objects of geometry — but how can there be a <separate> heaven and its parts, or anything else with movement? Similarly with the objects of optics and harmonics; there will be utterance and seeing over and above perceptible individual utterances and seeings. Clearly this is true of the other sensings and objects of sense too — why one rather than another? But if so, there will be <separate> animals too, if there are <separate> sensings.[6]

Aristotle's claim in the B.2 passage is that if the objects of the pure mathematical sciences are intermediates, then not only astronomy (and mechanics, though he does not mention it here) but also optics and harmonics will require intermediate objects. Since these four sciences are also branches of mathematics, they too should have intermediate substances for objects. He then pushes the accumulation further: there will also be intermediate sensations or senses (αἰσθήσεις, 997b23), and intermediate animals (b24). The M.2 argument explicitly recalls the B.2 passage and relies on (while leaving unstated) some of its premises.[7] It also extends the point about the impossibility of an intermediate heaven to anything with movement and adds *all* sensations and sensible things to the accumulation.

3. IS THIS A BAD ARGUMENT?

In both passages, Aristotle's move from (1) arithmetical and geometrical intermediates to (2) intermediate sensible things (utterances, heaven and its parts, seeings) to (3) all sensible objects, sensations, and animals creates the strong impression that his aim is to undermine his opponents' view simply by accumulating many kinds of intermediates. This is how Madigan 1999 represents the B.2 argument in his commentary (56), and Annas, too, interprets both arguments in this way (143). She objects to the move from (2) to (3): "Aristotle's Platonist here is a straw man", since while an expert in harmonics might say that he studies not actual sounds but ideal sounds, he "would certainly not think of 'the ideal sound' as a sound, or give it the logical behavior of one, as Aristotle tries to force him to do" (143). On this interpretation, the move from intermediate numbers and figures to intermediate seeings and utterances may be warranted,[8] but the move from the latter to intermediate senses and animals is mere polemics.

My aim is to show that there is more going on in these passages, and in particular that they reveal something about Aristotle's own account of mathematical objects. The first task is to show that this is for Aristotle a serious argument, in that while of course he does not accept intermediates, he is committed to the premise that moves from arithmetical and geometrical intermediates to intermediate utterances and seeings (that is, from 1 to 2), and to the premise that moves from the latter to intermediate senses and animals (from 2 to 3).

The move from (1) to (2) rests on the premise that the mixed sciences are proper mathematical sciences. For Aristotle, the sciences in question (astronomy, mechanics, harmonics, and optics) are under or subordinate to (ὑπό) the pure mathematical sciences: optics, mechanics, and astronomy are under geometry, and harmonics is under arithmetic (*Posterior Analytics* 1.13 78b35–9).[9] They are called 'mixed' because they study the mathematical properties of different kinds of sensible things. Astronomy studies the mathematical properties of the motion of sensible heavenly bodies, optics of sensible sights (visual phenomena), harmonics of sensible voices or utterances, and mechanics of the motion of bodies. What distinguishes these sciences from each other is the kind of sensible object that they study. What distinguishes them from the pure mathematical sciences is that while the objects of both may be said of an underlying subject (because the lines, figures, etc. they study are properties of sensible things), in the case of pure mathematics they are not studied "as being said of an underlying subject", while in the case of the mixed sciences they are (*Posterior Analytics* 1.13 79a7-10 with 1.27 87a33-4; see also the contrast between astronomy and the pure mathematical sciences at *Metaphysics* Λ.8 1073b5–8). [10] Or as Aristotle puts it in *Physics* 2.2, the mixed sciences "are in a way the converse of geometry. While geometry investigates natural lines but not *as* natural, optics investigates mathematical lines, but *as* natural, not *as* mathematical" (194a9-12). That is, while both geometry and optics consider the mathematical properties of sensible things, optics (and not geometry) considers these properties insofar as they belong to a specific subset of sensible objects: visual phenomena.[11] Similar contrasts can be made between the pure mathematical sciences and the other mixed sciences.

It is clear that Aristotle considers these to be genuinely mathematical sciences. As noted above, he refers to them as "the more natural branches of mathematics" (τὰ φυσικώτερα τῶν μαθημάτων, *Physics* 2.2 194a7–8). There are several other passages where this is clear. For example, in *Metaphysics* Λ.8, he writes of "one of the mathematical sciences which is most akin to philosophy — viz. that of astronomy" (1073b4-5), and refers to astronomers as mathematicians (1073b1-12) and in *Posterior Analytics* 1.14 he lists optics (ὀπτική) as one of the mathematical sciences (αἵ … μαθηματικαὶ τῶν ἐπιστημῶν, 79a18-20). [12] So Aristotle endorses the premise that begins the accumulation—the premise that moves from (1) pure mathematical intermediates to (2) mixed mathematical intermediates.

One might object on behalf of the Platonists that the move from (1) to (2) is gratuitous: even if the pure mathematical sciences require intermediates, the mixed mathematical sciences are simply the application of mathematics to sensibles. Hence there is no need for a new and special kind of intermediate.[13] But Aristotle makes it clear in N.3 why, as reasonable as this view of the mixed sciences might be, his opponents cannot adopt it. His targets claim that "branches of knowledge (αἱ ἐπιστῆμαι) cannot have <perceptible> things as their objects" (1090a27–8), and they make mathematical objects separate (κεχώρισται τὰ μαθηματικά, 1090a29). But if mathematical objects were separate (that is, a kind of substance distinct and somehow apart from sensibles), "their attributes would not apply to bodies" (1090a29–30). Since anyone who separates the objects of the pure mathematical sciences cannot explain "why, if numbers are in no way present in perceptible things, their attributes apply to perceptible things" (1090b3–5), they are not warranted in applying their separated numbers and geometrical objects to sensibles. They must therefore posit a new kind of entity to serve as objects of the mixed mathematical sciences — an entity that has the connection between the mathematical and the sensible already built into its nature.[14]

So much for the move from (1) to (2). The move from (2) to (3) seems the more objectionable one. We have seen that Annas objects to it on the grounds that Aristotle is inappropriately insisting that his opponents treat an intermediate utterance (φωνή) or seeing (ὄψις) just like a sensible utterance or seeings — that is, as an entity sensed by (intermediate) senses possessed by (intermediate) animals.

However, Aristotle is not insisting that these intermediates must be just like their sensible counterparts. His argument only requires that they be like them in certain relevant respects. Ideal though they are, they must share certain features with their individual sensible counterparts if they are to serve as objects of their respective sciences. On the view Aristotle is targeting, each science requires a special kind of substance — number-substances for arithmetic, point-, line-, and figure-substances for geometry, and (as Aristotle has argued) seeing-substances and utterance-substances for optics and harmonics, respectively. Since pure mathematics on this view is about ideal arithmetical and geometrical substances, these entities must, like their sensible counterparts, be divisible and combinable; further, the ideal numbers must be composed of units, the solid figures must be bounded by planes, the planes by lines, the lines by points, and so on.[15] That is, they must be capable of undergoing the many operations arithmeticians and geometers regularly perform. The same should also be true for intermediate seeings and utterances. Harmonics is about utterances, and Aristotle has shown that on the targeted view, these utterances should be intermediates. These ideal utterances may differ from their sensible counterparts by e.g. being perfect instances of a note. But if they also differ by being inaudible, then they are not utterances at all — in which case, harmonics turns out not to be about utterances.[16] As Alexander observes, "how is it possible for there to be certain visible things, the objects of optics, if they are not sensible? Or for there to be audible things, the objects of harmonics, if they are not sensible? For the essence of optics is to speak about things that are visible, and the essence of harmonics is to speak about things that are audible"(198, 2-6).[17]

But do audible ideal utterances entail the existence of ideal senses and animals? This move looks more objectionable than the previous one, but Aristotle has good reasons for making it.[18] Being audible is a capacity for being heard; hence it is a potentiality. In *Metaphysics* Θ.3, Aristotle argues that something cannot even potentially be the case if its actuality is impossible (Θ.3 1047a24-6); and in Θ.4 he argues that "it cannot be true to say 'this is capable of being but will not be'" (1047b3-5). Thus if something is audible, it must be possible for it to be heard. Since we sensible, perishable animals never in fact hear ideal utterances, the audibility of intermediate utterances would seem to require ideal hearers — or as Aristotle says, intermediate senses and animals. Given his commitment to his Θ.3-4 premise, the move from (2) to (3)—from intermediate utterances and seeings to intermediate senses and animals — is not mere polemics for Aristotle. His opponents can certainly reject the Θ.3-4 premise, though not without the cost Aristotle there argues this would entail, that is, that nothing — even e.g. the commensurability of the diagonal and side of a square — is incapable of being. This is a high cost, and it is not clear that paying it would be better than accepting the existence of intermediate senses and animals. Hence Aristotle has a strong argument here against anyone who posits intermediates for arithmetic and geometry.

4. INTERMEDIATE SENSIBLES AND ARISTOTLE'S OBJECTIONS

Since Aristotle is not just foisting absurdities on his opponents, it is worth carefully considering the nature of the objects he argues they ought to accept, as well as his objections to them. This may reveal something about how to understand his own statements about the nature of mathematical objects. Since — if he has not committed a significant error — what Aristotle finds objectionable about intermediates should not apply to his own mathematical objects, understanding his objections may cast serious doubt on certain interpretations of his view.

We can begin with the mixed mathematical intermediates. The first thing to notice is that these are quite different in nature from the arithmetical and geometrical intermediates. While the latter are non-perceptible, Aristotle refers to the mixed sciences' objects as sensibles intermediate (αἰσθητὰ μεταξὺ) between individual sensibles and forms (997b23); and in the M.2 passage he describes the harmonic and optical intermediates as "voice and sight" (φωνή τε καὶ ὄψις) respectively, both of which are sensible things.[19] He specifies that these will have to exist in addition to (παρά + accusative) "the sensible, i.e. individual, voices and sights" (1077a4-6).

Why must these intermediates be in some sense sensible? Because what makes each of the mixed sciences the science that it is, and not just the pure science to which it is subordinate, is the fact that it studies certain mathematical properties insofar as they belong to specific kinds of natural objects or processes.[20] What makes optics *optics* is that it studies lines insofar as they belong to sensible sights (visual phenomena). If it studied lines apart from sensible sights, then it would be not optics but geometry. Note that the claim is not that optics studies lines that just happen to be in sensible sights, so that the only difference between optics and geometry is the substratum in which those lines happen to be. It is rather that optics studies these lines insofar as they belong to sensible sights. That is, at least one sensible property of sights is part of the causal story an optical scientist tells when explaining e.g. the shape of a visual phenomenon like a halo. Hence the objects of optics, unlike the objects of geometry, retain at least one sensible property, and it is appropriate to refer to them as intermediate sensibles.

The sensible property that distinguishes the objects of the mixed sciences from the objects of arithmetic and geometry is motion. Mechanics, unlike solid geometry, studies the motion of bodies; astronomy studies the motion of the heavens and its parts; harmonics studies the relationships between certain sounds played or voiced in sequence (musical scales, melody);[21] and optics studies visual rays.[22] Further, it is the motion of the visible and audible objects themselves that brings sense-perception about. For example, "colour sets in movement (κινεῖ) what is transparent, e.g. the air, and that [viz.

the movement of the air], extending continuously from the object of the organ, sets the latter [viz. the organ] in movement (κινεῖται)" (*De Anima* 2.7 419a13-15)[23]; and the production of a sound is the "setting in movement a single mass of air which is continuous up to the organ of hearing" (2.8 420a3–4).[24] Hence if there are intermediates for the mixed sciences, these intermediates will be in motion.

Such intermediates will be ideal-sensible hybrids: they will be imperishable and perfectly precise, yet capable of undergoing motion — and as we have seen, Aristotle refers to them as intermediate sensibles. Lear 1982 supposes that "having to admit that ideal objects move" is simply "embarrassing" for Platonists, and (pointing to *Republic* 528a-b and 529c-d) he observes that "it is far from clear that Plato was embarrassed by this" (167 and n. 10). But these objects are more than just embarrassing. They push proponents of intermediates into what Owen 1970 and Vlastos 1981 call a 'two-level paradox' — that is, a conflict between an object's ideal and proper attributes. [25]

Imperishable sensible objects that undergo motion are not inherently paradoxical; Aristotle himself accepts some such objects, and some (most prominently Lear 1982) have argued that they are perfectly precise (e.g. that the stars are perfect spheres). However, this kind of object is paradoxical for his opponents, since according to their principles it will have to be both immovable and movable. It will have to be immovable because of its status as an ideal mathematical substance: such objects (intermediates) are supposed to be motionless (ἀκίνητα, A.6 987b16-17, B.2 998a14-15, M.2 1076b35). But it will have to be movable because the mixed sciences study, among other things, the motion of their objects (e.g. the motions of the heavens, sounds). The ideal and proper attributes of these intermediates are incompatible.

The reason Aristotle's opponents face a two-level paradox while he does not is that they make an assumption about the ontology of mathematical objects that he does not make. This is that the objects of the mathematical sciences are in fact unmoving. Aristotle instead takes mathematical objects to be sensibles considered *qua* unmoving (ᾗ ἀκίνητα, E.2 1026a9–10), i.e. considered without their motion and its associated properties. As Mignucci 1987 helpfully explains, for Aristotle, "immobility is not a positive property of mathematical objects — if it were so, mathematical objects would have properties which would be inconsistent with properties of physical bodies" (181).[26]

Aristotle makes this very point when he objects that it is "not reasonable" to suppose that such an object is unchanged; yet "for it to be changed is altogether impossible" (997b19-20, trans. Madigan).[27] Alexander explains that "the essence and nature of these things is bound up with such and such a kind of motion" (ἡ γὰρ οὐσία καὶ ἡ φύσις τούτων μετὰ τῆς τοιᾶσδε κινήσεως). Aristotle recalls this B.2 objection in M.2, when he protests that it does not seem possible that there is "a heaven and its parts — or indeed anything which has movement" — apart from the sensibles (παρὰ τὰ αἰσθητὰ). Notice that he is explicitly extending this objection to anything else with movement (ἄλλο ὁτιοῦν ἔχον κίνησιν, 1077a4) — that is, to the objects of any science of sensibles whatsoever. His opponents will need to posit intermediates not just for the heavens, seeings, and utterances, but for every other kind of sensible thing, too. As he says at 1077a6–8, "this is true of the other sensings (αἰσθήσεις) and objects of sense (αἰσθητά) too — why one rather than another?". So for example, zoology will be about intermediate animals and medicine will be about intermediate healthy things. Like the intermediate heaven and its parts, these objects must be sensible-ideal hybrids; and so like the intermediate heaven and its parts, their nature as ideal entities requires that they be immovable, while their sensible nature requires that they be moving or changeable. Thus they must, impossibly, be both moving and immovable.[28]

This indicates that when Aristotle argues in B.2 and M.2 that his opponents will need intermediates for the mixed sciences, he is doing more than just insisting that since the mixed sciences are mathematical, they require intermediates (997b16–18). He is also pointing to a reason such intermediate sensibles are needed on his opponents' own principles — a reason that will also require them to accept intermediates for every science of sensibles.

The reason is this: his opponents are committed to the view that mathematical truth requires immovable mathematical substances, yet they seem to have no adequate objects for the mixed sciences. The forms will not do, because these sciences study, among other things, the motion or change of their objects, while the forms are immovable and unchanging. Alexander makes note of this in his comments on B.2: if something is not "enmattered and by its own nature sensible" then it cannot be in motion (ἀδύνατον γὰρ κινεῖσθαι τὸ μὴ ὂν ἔνυλον καὶ τῇ αὑτοῦ φύσει αἰσθητόν, 198, 13–14); so it is an absurdity (ἄτοπον) to hold "that there is some Idea of heaven, Heaven Itself, and of the sun, Sun itself; for how is it possible to conceive of any of these as immovable (ἀκίνητον)?" (198, 14-16). Since forms are essentially immovable and the heaven and sun are essentially movable, they are by their very natures incompatible. We can call this the "movability problem". But neither can sensibles as such be the objects of any science — a point of agreement for Aristotle and his opponents.[29] As we have seen, according to Aristotle's account, his opponents posit intermediates for geometry and arithmetic because neither forms nor sensibles can serve as objects for these sciences: the uniqueness problem rules out forms, while perishability and imperfection rule out sensibles. They ought then to posit intermediates to secure the truth of the mixed sciences, since the uniqueness problem and the movability problem rule out forms, while perishability (and perhaps also imperfection) rule out sensibles. In fact, they should posit them for zoology, medicine, and the other sciences as well, since the movability problem again rules out forms, and the perishability of sensibles rules them out as objects for these sciences, too.

In short, two of the same problems — the immovability of forms and the perishability of sensibles — that warrant positing intermediates for mixed mathematical sciences also warrant them for unmixed sciences of sensibles like zoology and medicine. When Aristotle extends the accumulation to senses and animals, he is not simply showing that his opponents will need to bite the bullet and accept these objects for the sake of the mixed mathematical sciences. He is also showing that intermediates are required for *all* sciences studying sensibles. Since the ideal-sensible nature of all such intermediates renders them paradoxical, the problem is not just that the accumulation is embarrassing; it is also, perhaps more importantly, that the accumulated objects are impossible.

5. WHAT THIS CAN TELL US ABOUT ARISTOTLE'S OWN VIEW

I have argued that the B.2 and M.2 passages constitute a serious argument, and that Aristotle is committed to the key premises that produce the accumulation. If this is correct, then he ought to try to avoid these objections in his own account of mathematical objects.

On one common line of interpretation, Aristotle takes mathematicians' statements to be about entities distinct in kind from sensible objects — entities that are in some sense mind-dependent. Alexander is an early proponent of this interpretation. He writes: "mathematical objects do not subsist independently, but by thought (ἐπινοίᾳ); for after the matter and the motion have been separated from enmattered things, the things according to which and with which mathematical objects have their subsistence, these objects are left" (*On Aristotle* Metaphysics, 52.15-18). Contemporary commentators who take this line suggest that mathematical objects are somehow tied to or constrained by the sensible world, but distinct from sensible things and mind-dependent.

For example, Modrak 2001 argues that "the mathematician realizes a potentiality in thought that is not realized concretely" (121), and that "the arithmetical unit is actualized as an object of thought" (123). On this view, mathematical objects exist potentially but never actually in the sensible world, because they are always only "imperfectly exemplified in physical objects" (120). They can only be actualized by the mathematician's thinking: an actual mathematical object is a conceptualization (122). Since these ideal objects are actualizations of potentialities in the sensible world, Modrak denies that they are "mere projections of the mathematician's mind" (122). Nevertheless, all actual mathematical objects are "dependent upon the way humans conceptualize the world" (123). Along similar lines, Halper 1989 insists that mathematical objects do not exist only in the intellect, since they exist potentially in the sensible world. However, he argues that they only exist actually in the intellect (265–6, 268–9).[30] But the mathematician studies the actual objects of mathematics: Aristotle insists that ἐπιστήμη is always of what is prior (ἀεὶ … περὶ τὰ πρότερα ἡ ἐπιστήμη, *Metaphysics* M.2 1076b35-6), and he devotes Θ.8–9 to showing that the actual is prior in every way to the potential. If then mathematical objects are actualized only in thought, it follows that all of the mathematician's proofs and statements are about perfect objects that exist only in thought.[31]

On this view, Aristotle and his opponents agree that the mathematician does not study objects in the sensible world; they disagree only over the ontological status of the mathematicians' objects. While Aristotle's opponents make them ideal, thought-independent substances, Aristotle makes them ideal, thought-dependent non-substances. But if this is indeed Aristotle's view, then he is vulnerable to much the same objection he levels against his opponents. This is because, as we have seen, the mixed sciences are genuinely mathematical for Aristotle. As he states in the *Physics* 2.2 passage, optics studies mathematical lines *qua* natural — that is, optics studies the same objects as geometry (lines and figures), only it studies them insofar as they belong to certain sensible things (visual phenomena).[32] If (as the interpretations in question hold) the objects of geometry do not exist in actuality in the sensible world, but rather only as ideal objects of thought, then optics will be the study of ideal thought-objects insofar as they belong to visual phenomena, i.e. *qua* natural. Similarly, the objects of harmonics, astronomy, and mechanics will be ideal thought-objects *qua* natural.

We can see why this is a problem if we consider any of the mixed sciences. We can take astronomy as our example. One of astronomy's principal concerns is to investigate the circular motion of its objects. On the interpretation in question, this means that astronomy studies circles *qua* natural, and these circles are themselves ideal objects existing only in thought. Now, Aristotle is clear that what exists only in thought cannot undergo locomotion, except incidentally (as the soul is moved when the body is moved; *De Anima* 1.3). This is because locomotion is change of place, so that what undergoes locomotion must have place. While a thought-object may have a location (where the soul of the thinker is located, and so at the same location as the body), it does not have place. Place is the innermost motionless boundary of what contains a body (*Physics* 4.4), and objects existing only in thought are not bodies.[33] So if geometrical points, lines, and circles exist only in the minds of mathematicians, the astronomer — who studies geometrical points, lines, and circles *qua* natural — is studying something *per se* immovable *qua per se* movable. In the first place, this is asking too much of the *qua*, which can only isolate a property (or set of properties) an object already has.[34] Indeed, the *qua* locution is closely associated with ἀφαίρεσις — subtraction — while on this interpretation, it works like addition (πρόσθεσις).[35] But even if the *qua* could do this work, the astronomer would be left with an impossible object: a *per se* immovable thought-object that is *per se* movable.

In short, if Aristotle's mathematical objects exist only in thought, then he and his Platonic opponents have almost the same problem with the objects of the mixed mathematical sciences. Even if they are derived from the sensible world, in that they are actualizations of what is only ever imperfectly expressed in sensible objects, the fact remains that all mathematical statements and proofs will be about these actualizations — these perfect entities existing only in the mind of the mathematician. Their nature as thought-objects will require them to be immovable; but their status as objects of the mixed sciences requires that they be moving. Thus like the Platonists Aristotle targets in B.2 and M.2, his own view of the objects of the mixed sciences would, paradoxically, have them be both immovable and movable. Of course it is possible that Aristotle has committed this error. But since there are other plausible interpretations available on which he does not, this tips the scale in their favor.[36]

6. CONCLUSION

My aim has been to show that it is fruitful to consider Aristotle's own reasoning about mathematical intermediates. The examination has revealed first that while Aristotle's targets' ontology requires intermediates for the pure mathematical sciences because of Annas' uniqueness problem, there is an additional reason it requires them for the mixed mathematical sciences, and indeed for all sciences of sensibles. This is the movability problem: since every science of sensibles explains some kind of movability or change, and since forms are essentially immovable, forms cannot be the objects of any science of sensibles. Second, I have argued that Aristotle is committed to the premises of his B.2 and M.2 argument against these objects, and to the argument's moves from (1) arithmetical and geometrical intermediates to (2) intermediate sensible things to (3) all sensible objects, sensations, and animals. If this is correct, then his objections to Platonic intermediates are more than mere polemics. Since the problem his B.2 and M.2 argument identifies — that these objects must, impossibly, be both immovable and movable — is one he himself formulates and takes seriously, it is unlikely that his own account of mathematical objects would fall prey to it. This in turn casts doubt on a common interpretation of his philosophy of mathematics, as it would have him running into just this problem. Thus a further result of the examination is that it reveals something useful for understanding Aristotle's own view about the ontological status of mathematical objects.

BIBLIOGRAPHY

Annas 1975: Julia Annas, 'On the "Intermediates"', *Archiv fur Geschichte der Philosophie* 57 (1975), 146-66.

Annas 1976: Julia Annas, *Aristotle's* Metaphysics *Books M and N*, Clarendon Press, Oxford 1976.

Arsen 2012: Hera S. Arsen, 'A Case for the Utility of the Mathematical Intermediates', *Philosophia Mathematica* 20 (2012), 200-223.

Barker 1989: Andrew Barker, *Greek Musical Writings*, Vol. 2, Cambridge University Press, Cambridge 1989.

Barnes 1984: Jonathan Barnes, *The Complete Works of Aristotle: The Revised Oxford Translation*, Princeton University Press, Princeton 1984.

Burnyeat 2000: Myles Burnyeat, 'Plato on Why Mathematics is Good for the Soul', in Timothy Smiley (ed.), *Mathematics and Necessity: Essays in the History of Philosophy*, *Proceedings of the British Academy* 103 (2000), 1-81.

Burnyeat 2005: Myles Burnyeat, 'Archytas and Optics', *Science in Context* 18 (2005), 35-53.

Cleary 1985: John Cleary, 'On the Terminology of "Abstraction" in Aristotle', *Phronesis* 30 (1985), 13-45.

Cook Wilson 1904: J. Cook Wilson, 'On the Platonist Doctrine of the ἀσύμβλητοι ἀριθμοί', *Classical Review* 18 (1904), 247-60.

Distelzweig 2013: Peter M. Distelzweig, 'The Intersection of the Mathematical and Natural Sciences: The Subordinate Sciences in Aristotle', *Apeiron* 46 (2013), 85-105.

Dooley & Madigan 1992: Dooley, William E. & Arthur Madigan, trans., *Alexander of Aphrodisias: On Aristotle's* Metaphysics *2 & 3*, Duckworth, London 1992.

Hussey 1991: Edward Hussey, 'Aristotle on Mathematical Objects', *Apeiron* 24 (1991), 105-33.

Katz 2014: Emily Katz, 'An Absurd Accumulation: *Metaphysics* M.2 1076b11-36', *Phronesis* 59 (2014), 343-68.

Lear 1982: Jonathan Lear, 'Aristotle's Philosophy of Mathematics', *The Philosophical Review* 91 (1982), 161-92.

Madigan 1999: Arthur Madigan, *Aristotle:* Metaphysics *Books B and K 1-2*, Oxford University Press, Oxford 1999.

Mignucci 1987: Mario Mignucci, 'Aristotle's Arithmetic', in Andreas Graeser (ed.), *Mathematics and Metaphysics in Aristotle*, Paul Haupt Berne, Bern 1987, 175-211.

Modrak 2001: D. K. W Modrak, *Aristotle's Theory of Language and Meaning*, Cambridge University Press, Cambridge 2001.

Mueller 1990: Ian Mueller, 'Aristotle's Doctrine of Abstraction in the Commentators', in Richard Sorabji (ed.), *Aristotle Transformed: the Ancient Commentators and their Influence*, Duckworth, London 1990, 463-80.

Owen 1968: G.E.L. Owen, 'Dialectic and Eristic in the Treatment of Forms', in G.E.L Owen (ed.), *Aristotle on Dialectic: The* Topics, Clarendon Press, Oxford 1968, 103-25.

Vlastos 1981: Gregory Vlastos, *Platonic Studies*, 2nd edition, Princeton University Press, Princeton 1981.

White 1993: Michael J. White, 'The Metaphysical Location of Aristotle's Μαθηματικά', *Phronesis* 38 (1993), 166-82.

NOTES

1 Arsen 2012 is an exception.

2 By 'Platonists' I mean the opponents Aristotle is targeting here, namely those who posit forms and intermediates. I consider Aristotle's reasoning independently of the question whether Plato himself posits intermediates.

3 *Metaphysics* B.2 997b35-998a3, K.1 1059b10-12, N.3 1090a35-b1

4 See also Cook Wilson 1904, 249–51, for an earlier description of the same difficulty.

5 All translations of Aristotle (except for *Metaphysics* M-N) are from Barnes 1984.

6 All translations of *Metaphysics* M-N are from Annas 1976.

7 Annas 1976 supposes that in M.2 the argument is "tacitly generalized over all ideal mathematical objects" (142). I do not see evidence of this generalization,

and find it implausible given that Aristotle is speaking here as in B.2 about the objects of the mathematical sciences.

8 Annas finds fault with this part of the argument, too. She suggests that a Platonist might respond that he has no problem with saying that there are ideal objects for all of the sciences: these are the forms. However, it seems unlikely any proponent of Platonic forms would wish to respond in this way, because it would imply that the "separate heaven" that is the object of astronomy is a form. We have seen that this is problematic because the heaven that the astronomer studies (which includes the sun, the moon, and the many stars) is in motion, and hence this form would have to be in motion. The Platonists would presumably recognize that a moving form is not possible.

9 This passage does not specify that astronomy is under geometry, though that Aristotle so classifies it is clear from his descriptions of what astronomy studies. See also *Posterior Analytics* 1.7 75b14–17.

10 In *Metaphysics* A.9, Aristotle argues that while "things in this world (e.g. harmony) are ratios of numbers", this does not support the view that there are forms that are numbers—that is, ideal number substances. For a harmony (συμφωνία) and each of the other kinds of ratios in sensible things, there is always "some one class of things of which they are ratios". He refers to this "some one thing" as "the matter" (*Metaph.* A.9 991b13-15).

11 Thus to say that optics studies mathematical lines *qua* natural is not to say that it studies all the natural properties of natural lines. Burnyeat 2005 explains that although the lines studied by optics are studied *qua* natural, they remain mathematical. Hence "[y]ou cannot legitimately infer that if the rays move, or have degrees of strength, they are corporeal and therefore have standard properties of physical bodies such as weight and thickness" (36).

12 See Distelzweig 2013 90-91 for additional passages and argument showing that these are indeed mathematical sciences, for Aristotle.

13 Thanks to Lloyd Gerson for raising this concern.

14 We will see (section 4) why such objects are problematic.

15 For this argument, see Katz 2014 354 and 354 n. 12. Arsen 2012 argues that for the same reason, they must be relational (209).

16 One might object that in *Republic* 7, Plato has Socrates propose a new kind of harmonics that would study the consonance of numbers independently of auditory experience (530d–531c). (See Burnyeat 2000 52–3 for an interpretation of this passage.) Perhaps then Platonists would reject the claim that the ideal utterances in question are audible. However, like the new astronomy described at *Republic* 529c–530c, this science of inaudible consonance is not the mixed science Aristotle is targeting. The mixed science is the one Socrates says involves "measuring audible consonances and sounds against one another" (531a). Hence even if, for Platonists, there is another harmonics of inaudibles, the problem with the objects of the harmonics of audibles remains.

17 All translations of Alexander are from Dooley and Madigan 1992.

18 Annas takes the move to rest on the premise that ideal sounds are *produced* in a manner exactly like sensible sounds: "from ideal throats of ideal people" (143). However, since Aristotle speaks of sense-objects and sensings, it is more natural to read him in the way I suggest, that is, as concerned not with how these ideal sounds are produced, but rather with how they are perceived. Further, even if we include Annas' premise, so that Aristotle is *also* thinking of how these objects of harmonics are produced, this is not a terrible argument. Aristotle is speaking not of sound (ψόφος) (as Annas has it) but of utterance or what is voiced (φωνή). In *De Anima* 2.8 he distinguishes between these two kinds of audible objects and specifies that what is voiced "is a kind of sound characteristic of what has soul in it; nothing that is without soul utters voice" (ἡ δὲ φωνὴ ψόφος τίς ἐστιν ἐμψύχου· τῶν γὰρ ἀψύχων οὐθὲν φωνεῖ, 420b5–6). Hence there is a tight connection between what is voiced and what has a soul (i.e. animals).

19 The claim here that there are intermediate sensibles is different from the earlier claim (997b3-12) that forms are just eternal sensibles.

20 For detailed discussion, see Distelzweig 2013, 94-100.

21 Aristotle states that harmony (ἁρμονία) is about "magnitudes which have motion and position" (τῶν μεγεθῶν ἐν τοῖς ἔχουσι κίνησιν καὶ θέσιν, *De Anima* 1.4 408a5-7), and sounds are motions of masses of air (*De Anima* 2.8). In his *Elementa harmonica*, Aristoxenus, a pupil of Aristotle, defines harmonics as "the science which deals with all melody, and enquires how the voice naturally places intervals as it is tensed and relaxed. For we assert that the voice has a natural way of moving, and does not place intervals haphazardly" (Barker 1989, 149). He argues that the conditions for understanding music are that we must "perceive what is coming to be and remember what has come to be" (Barker 1989, 155). And a Platonist taking seriously *Timaeus* 80a-b would also think that harmonics studies (among other things) the motion of its objects (sounds). Plato has Timaeus explain pitch in terms of the speed at which sounds travel, and harmony and lack of harmony in terms of the motion produced by slow and fast sounds as they move toward the auditor.

22 The visual ray (ὄψις or ἀκτίς) for Aristotle is the line of sight from the eye to the object seen. This is what "arrives at" (ἀφικνεῖται ἡ ὄψις, *De Caelo* 2.8 290a21) the visible object, and what is reflected by the air (under certain atmospheric conditions) and smooth surfaces (ἡ ὄψις ἀνακλᾶται *Meteorology* 3.2 372a29–31), producing visual phenomena like halos and rainbows. He is critical of the account given in the *Timaeus*, according to which the visual ray is a body (σῶμα), and specifically a kind of internal fire flowing out from the eyes (45b4-8) that coalesces with external fire (typically daylight) (45c2-5). Aristotle sharply criticizes this view in *Sense and Sensibilia*

2: “It is, to state the matter generally, an irrational notion that the eye should see in virtue of something issuing from it; that the visual ray should extend itself all the way to the stars, or else go out merely to a certain point, and there coalesce, as some say, with rays which proceed from the object” (438a25–7).

23 This is not to say that for Aristotle light (φωτός) travels (φέρω). Indeed he denies this (*De Anima* 2.7 418b20–4). Light is the actuality of the potentially transparent (which is excited to actuality by e.g. fire).

24 This is also reflected in the third century *Sectio Canonis* (possibly Euclid’s work). In the *Sectio*’s introduction, the author writes: “If there were stillness and no movement there would be silence”, and that “some notes must be higher, since they are composed of closer packed and more numerous movements, and others lower, since they are composed of movements more widely spaced and less numerous” (Barker 1989, 191-2).

25 Vlastos discusses this paradox as it applies to Platonic forms. The paradox arises because of a conflict between the ideal and proper attributes of forms—that is, properties the form F has *qua* form (ideal) or *qua* F. As Vlastos puts it, “The Idea of *F* is *P*” is “true if *P* is predicated of ‘the Idea *qua* Idea,’ (…) [but] false if predicated of it ‘*qua F*’” (323). Vlastos notes that this is not really a paradox for Aristotle, since he sees that it is one thing to assert a predicate of F *qua* form and quite another to assert it of F *qua* F. But it remains a paradox for anyone who either has not acknowledged this distinction, or whose other views prevent him from doing so consistently. Vlastos argues that Aristotle believes that Plato is such a one (323–4).

26 He continues: “Mathematical objects do not move, not because they are unaffected by movement, but because movement is left out of consideration” (181).” Mignucci is explaining (180–1) why there is no inconsistency in Aristotle’s view that the objects of geometry are the shapes or limits of sensible things (for example in *Physics* B.2). But the same point explains why there is no inconsistency in his view that the objects of the mixed mathematical sciences are sensibles.

27 See Madigan (56).

28 Later in B.2, Aristotle makes the same objection to a different kind of ideal-sensible intermediate: arithmetical or geometrical intermediates located in sensible objects. Since these intermediates are in sensibles, which are moving, they will not be immovable; but according to the theory, intermediates are supposed to be immovable (998a14–15). (Aristotle claims that certain thinkers have adopted this problematic view (998a7–9). Perhaps they did so in response to the problem Aristotle raises in N.3. (See the objection to the move from (1) to (2), in section 3 above.)

29 Plato argues that sensible things are only opinable, and not knowable, because they are between what purely is and what in every way is not. The many beautiful visible things are in a way beautiful and in a way not beautiful; and the many doubles are in a way doubles, but also in a way halves (*Republic* 476a9-480a13). Knowledge can only be of “the things themselves that are always the same in every respect” (*Republic* 5 478e7--480a13; see also e.g. *Phaedo* 74a9-77a5). Aristotle agrees that the objects of knowledge must be imperishable (e.g. *Nicomachean Ethics* 6.3 1139b19-24), and in *Metaphysics* B.2 he seems to acknowledge that geodesy and astronomy cannot study sensible magnitudes and the sensible heavens respectively, since sensible magnitudes are perishable (so that geodesy would then perish along with its objects) and sensible objects and processes seem not to be like or the same as (ὅμοιον, τὸ αὐτό) the objects the geometer or astronomer describes (997b32-998a6). However, I do not take the B.2 passage (nor K.1 1059b7–12, the other oft--cited passage) to rule out sensibles as the proper objects of these sciences. I rather understand B.2 and K.1 in light of Aristotle’s later insistence, in M.3, that mathematics and other sciences are about sensibles *qua* a certain subset of their properties. Hence what he is denying in B.2 and K.1 is only that these sciences are of sensibles *qua* all (and in the case of geometry, any) of their sensible properties.

30 See also White 1993, 179–81. All of these commentators are careful not to claim that mathematical objects are utterly disconnected from the sensible world. This makes their view unlike what Mueller 1990 calls the “mentalistic” interpretation, which he associates with “at least the majority of the [ancient] commentators” (465). This modern view is rather that mathematical objects are somehow potentially in the sensible world. However, all those who take this line agree that this is a special kind of potentiality, inasmuch as it cannot be actualized in the sensible world, but rather only in the mind of the mathematician.

31 These interpretations appear to be prompted by (1) *Metaphysics* B.2 997b34–998a6 and K.1 1059b7–12, where Aristotle states that mathematicians do not treat of sensible things, and (2) *Metaphysics* Θ.9 1051a29-32, where Aristotle states that “the potentially existing [geometrical] relations are discovered by being brought to actuality (the reason being that understanding is an actuality)” (trans. Ross, slightly modified). (1) As for the first set of passages: while I do not quite follow Lear 1982 in asserting that in B.2 “it is an imagined Platonist speaking, and not Aristotle” (176), I agree that these passages do not count against the view that for Aristotle, mathematicians study sensible things. This is because Aristotle claims that mathematicians in fact study sensibles; they just do not study them *qua* sensible (*Metaphysics* M.2 0178a2-5, *Physics* 2.2 193b22-5 with b31–3). So we can understand an implied “*qua* sensible” in the B.2 and K.1 statements. (2) In the Θ.9 passage, Aristotle does not state that the mathematician actualizes geometrical objects by means of an intellectual working-up of the sensible (as Modrak and Halper have it). Rather, the passage simply describes how the geometer works: she discovers geometrical relations (e.g. symmetry, similarity, parallelism, etc.) by dividing figures and so producing new ones (e.g. by dividing the line AB at the point C, she produces or actualizes the lines AC and CB).

32 See also e.g. *Posterior Analytics* 1.12, 77b1–2, where Aristotle states that optical things (τὰ ὀπτικά) are "proved from the same things as geometry" (ἐκ τῶν αὐτῶν δείκνυται τῇ γεωμετρίᾳ).

33 This is why the soul is in place only accidentally (*Physics* 4.5 212b11-12).

34 Lear aptly describes it as a "predicate filter". A filter does not add predicates; it removes them.

35 For a lucid account of Aristotelian ἀφαίρεσις, see Cleary 1985.

36 For example, Lear 1982's view that the geometer "considers genuine properties of objects, in particular, geometrical properties actually possessed by physical objects" (186). Lear, like the above-mentioned interpreters, takes mathematical objects to exist only in the mind of the mathematician; he calls them harmless (172) and useful (188) fictions. However, he distinguishes between these objects—the objects to which terms like 'triangle' refer—and the truthmakers for mathematical statements. He takes the latter to be geometrical properties perfectly instantiated in the sensible world (e.g. the spherical shape of the stars) (169). I have a different interpretation of Aristotle's philosophy of mathematics; but it should be noted that Lear's is not vulnerable in the same way that the above-mentioned views are.