

# Plato's Intellectual Development and Visual Thinking

**Susanna Saracco**

[susanna.saracco@gmail.com](mailto:susanna.saracco@gmail.com)

<https://orcid.org/0000-0002-8655-7912>

## ABSTRACT

Plato has devised texts which call the readers to collaborate cognitively with them. An important epistemic stimulation is the schematization, the line segment, which summarizes Plato's idea of intellectual development. In this research, visual thinking will help us to make the most of the Platonic invitation to investigate further cognitive growth. It will be analyzed how visual discoveries are rendered possible by mental number lines, realizing the epistemological importance of visualization. Thanks to visualization, structuralism will be grasped. It will reveal a connection with Plato's philosophy which suggests a novel elaboration of the Platonic concept of intellectual growth.

Keywords: Higher-Order Pedagogy; Theoretical Childhood; Theoretical Adulthood; Visual Thinking; Mental Number Lines; Structuralism.

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## 1. PLATO AND THE RATIONAL ENGAGEMENT OF HIS READERS

The present research is rooted into my work on Plato and intellectual development (Saracco 2017). There, I analyzed a crucial passage of the Platonic dialogues: the passage of the sixth book of the *Republic* (R. VI 509d-511) in which Plato explains what are, for him, the stages of intellectual development of the human being and what are the objects of knowledge pertinent to each phase of cognition. Plato schematizes his idea of intellectual progress using a line segment divided into four subsections: two of them correspond to phases in which our knowledge is still connected to the sensible realm and the other two sectors indicate a kind of knowledge which is pertinent to the intelligible realm.

My attention was captivated by the moment in which Plato, summarizing his idea of cognitive progress, tells his readers that there is much more to know about the subject than what had been discussed so far with Glaucon (R. VII 534a):

But as for the ratios between the things these are set over and the division of either the opinable or the intelligible section into two, let's pass them by, Glaucon, lest they involve us in arguments many times longer than the ones we have already gone through. (My emphasis)

Foley (2008, 23), commenting the previous excerpt from the *Republic*, emphasizes:

the passage shows that Plato is not willing to set forth his views on the further complexities that have emerged. It is a task that he *intentionally* leaves for his readers, revealing that his final assessment of the role of the divided line is to

*force a thoughtful reader to transcend the text.* One significant aspect of the divided line is exactly that Plato refuses to explain its point. (Foley 2008, 23. My emphasis)

Foley's words reveal a crucial insight: Plato's text is a stimulus for a rational investigation which is not meant to *end* in the written words of his dialogues. Plato asks his readers to participate actively with the text. This participation is not meant to be a simple approval or criticism of the words of the philosopher; rather, this call for collaboration is designed to "*force a thoughtful reader to transcend the text*" (Foley 2008, 23. My emphasis). Plato, presenting in the *Republic* his schematization of intellectual development, in connection with the objects of investigation that human reason can grasp, *tells* his readers that there is more to discover on the subject, and this is something that *they* have to do. In saying this, Plato *calls for a collaboration between writer and reader.* Plato has not written a textbook whose content can merely be summarized by the readers. He has created a text to which they are required to respond and the act of responding to the text is as important as the text itself: the two of them together complete Plato's task. Plato does not want to convey a static description of how things are. He has created a text that calls out for completion by the readers' further contributions. This does not mean that Plato's words are incomplete in the sense that they communicate thoughts which have not yet reached a good degree of elaboration. On the contrary, it means that the words written by Plato are so well mastered by their author that they are able to stimulate the reader to overcome them, as Foley was highlighting. Plato's texts are not only composed by words which have the goal of expressing the thinking of their author but they also comprise the thinking of their users.

Through the dialogues, Plato is inviting us to reflect on *our* cognitive resources to develop them autonomously. He says this explicitly in the *Meno*:

As the whole nature is akin, and the soul has learned everything, nothing prevents a man, after recalling one thing only—a process men call learning—discovering everything else for himself, if he is brave and does not tire of the search, for searching and learning are, as a whole, recollection. (*Men.* 81c-d)

It is useful to read these lines together with an excerpt from the *Phaedrus*, where Socrates is reporting a dialogue about the art of writing which takes place between Thamus and Theuth:

O most expert Theuth, one man can give birth to the elements of an art, but only another can judge how they can benefit or harm those who will use them. And now, since you are the father of writing, your affection for it has made you describe its effects as the opposite of what they really are. In fact, it will introduce forgetfulness into the soul of those who learn it: they will not practice using their memory because they will put their trust in writing, which is external and depends on signs that belong to others, instead of trying to remember from the inside, completely on their own. You have not discovered a potion for remembering, but for reminding; you provide your students with the appearance of wisdom, not with its reality. Your invention will enable them to hear many things without being properly taught, and they will imagine that they have come to know much while for the most part they know

nothing. And they will be difficult to get along with, since they will merely appear to be wise instead of really being so. (*Phdr.* 275a-b)

Let us consider this passage in connection with the passage of the *Meno* cited above: in the *Meno* Plato tells us that learning is a process of “recollection” (*Men.* 81d) and in the *Phaedrus* we read that the written words will not help us to remember but they can only be used as *reminders* because they do not lead to ourselves but they rather depend on signs that “belong to others” (*Phdr.* 275a). In the *Phaedrus* Plato explicitly connects the process of learning with remembering something that is *inside* us: what is inside us makes us remember, recollect, a wisdom that is merely reminded by the written words.

It seems unlikely that the author of these passages would conceive of his own written words as the final destination of knowledge, but rather as a stimulus to reach that destination, which is internal to us. Thus, the Platonic words are only a *reminder* of the necessity of looking for knowledge where the answers to the dialogical questions come from, *inside* us, in the organ capable of remembering which is, for Plato, the soul and its main component, the reason. Consistently, Plato’s dialogues do not end with the thoughts of the author and the words, the *reminders*, that he has selected to convey them, but they are enriched by the multitude of rational memories prompted by the autonomous investigations of Plato’s readers.

The courage of recognizing the existence of an intellectual dimension in which what we have learned to consider certain becomes criticizable, losing its stability, is the necessary premise to reconstruct creatively a truth, which is far from the shadows of what merely appears as true, as Mattéi makes us unders-

tand in the following quote, distinguishing “two sorts of spectacle lovers” (Mattéi 1988, 79. My emphasis):

The first are the crowd and the sophists who unreservedly dedicate themselves to the sensible beauty of colors, forms and voices. As Socrates puts it to Glaucon: ‘those who love to watch’ (φιλοθεᾶμονες) and ‘those who love to listen’ (φιλήκοοι; R. 475d2) remain the *prisoners of appearances* even if they show an unconscious desire for a higher kind of knowledge. In front of them, ‘those who love to know’-*the philosophers*-are in search of the luminous theater of *truth beyond the shadow play*. Like the pure souls released from their bodies and contemplating the vast plain of Truth, and like the initiate in Eros’ mysteries contemplating the boundless ocean of the Beautiful, ‘the genuine philosophers are those who are in love with the spectacle of the truth’ (R. 475e).

Here Mattéi highlights that the spectacle created by Plato must not be seen as something constructed to be passively watched and it is not the final destination of the intellectual growth of the reader. If we confuse a means of rational growth with the final goal of this process, we are condemned to live in an epistemic realm in which the shadows are for us the reality. In this cognitive dimension we will never know the truth. If we recognize that Plato’s words compose a succession of epistemic stimulations devised to encourage rational evolution, whose meaning requires to be completed by the critical and creative contributions of his readers, we allow the words of Plato to perform the real show they were invented for, the show in which the absolute protagonist is human reason.

## 1.1 RATIONAL ENGAGEMENT AND HIGHER-ORDER PEDAGOGY

The dialogical character of Plato’s work is opposite to the will of indoctrinating or just instructing the readers. Plato *chose* to write *dialogues* and this choice is not only a formal but also a philosophical choice: Plato wants to stimulate an active participation of his readers which goes beyond the accidental criticism of the written words, which can take place whenever a text is read. In fact, as we have just seen, when Plato in the *Republic*, has presented his idea of what intellectual development is, he states explicitly that there is more to discover on the subject, but he does not tell his readers *how* they should do it. The modes of collaboration between writer and reader advocated by Plato are not predetermined by the philosopher. Plato’s readers can choose to criticize radically his philosophical system or they can choose to accept its basics. Plato interacts dialogically with his readers, asking them explicitly to transcend the text (Foley 2008, 23. Also cf. *Phaedrus*, 275 a-b) to complete it with their contributions. This Platonic request is at the base of the *higher-order pedagogy* that permeates the dialogues, where the role of the readers is not flattened to that of students who can merely absorb the content proposed by their teacher. Plato’s readers are invited to become active creators of the philosophical message. This invitation has not to be considered as a consequence of a lack in Plato’s argumentative ability. On the contrary, as we have just seen, the philosopher is able to stimulate his readers with explicit requests.<sup>1</sup>

For Plato, education has crucial importance. In fact, the philosopher is well aware of the fact that the human rational nature can diverge from its positive capabilities, when its direction

is determined by messages that appeal simply to appetite. This intuition is itself extraordinary for its modernity. But what renders the Platonic rational pedagogy extraordinary is its character: Plato explicitly says to his readers that they have to find the truth by themselves, using what they are reading only as reminder of the rational power that they possess (*Phaedrus*, 275 a-b). Plato's is a kind of higher-order pedagogy in which the readers are not the passive receptors of a content but they discover themselves as authors of the content.

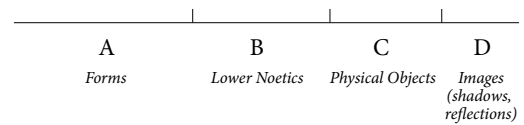
The dialogue between Plato and his readers takes place via the written words of his texts, that allow the continuation of the cognitive exchange between the philosopher's rational heritage and his reader's intellect. The dialogical interaction with the readers, and the consequent free development of their thinking abilities, does not mean that the Platonic philosophy can be developed in any way. The intellectual stimulation of Plato's words consists in the exhortation to contribute in an original and creative way to the development of what Plato thinks that knowledge is. Plato tells his readers clearly what his idea of knowledge is: the highest point of intellectual development is reached when we are able to abandon the empirical completely to reach the purely intelligible. Only when our rationality is disentangled from the distracting stimuli which come from the tangible realm, we are able to grasp the purely intelligible truth. Nonetheless, the individual contributions of Plato's readers can mould the concept of Platonic knowledge into the shape their intellect suggests. Furthermore, it remains possible at any point for Plato's readers to use their rational capabilities, sharpened through the texts written by the philosopher, to criticize his conception of knowledge, abandoning in this way Plato's philosophical system. My work does not go in this direction. I have chosen to

respond to the Platonic intellectual stimulation, proposing a new theoretical framework for engaging with Plato's dialogues.

### 1.1.1 PLATO'S HIGHER-ORDER PEDAGOGY: MY RESPONSE.

I am going to present the basics of the new theoretical lens that I have elaborated as response to the Platonic request to collaborate with his text. As we will see, my reflections on Plato's schematic representation of intellectual development, will be enriched by new considerations on the role of visual thinking and visual discovery in Plato. We will analyze how the schematization of the line segment aids visual discoveries and how these discoveries relate the line segment with mathematics.

I have chosen to accept the core of Platonic philosophy and I have decided to engage with his words, using them for an investigation in line with his philosophical system. At the centre of my engagement with Plato's words there is the account of human intellectual development presented in the *Republic* (*R.* VI 509d-511), schematized using a line segment divided into four subsections:



This is the rendition, chosen by Foley, of Plato's discussion of the progress of the cognitive capacities of the individual. Each object indicated in the line segment above can be apprehended thanks to a rational faculty correspondent to it. (Foley 2008, 1) The subsection A corresponds to Understanding, *noēsis*. At this

stage of intellectual development the individual is able to apprehend the Forms. The subsection B is Thought, *dianoia*. In this phase of rational evolution the person begins his investigation of the mathematical objects, intellectually inferior to the Forms. The subsection C is Belief, *pistis*, which gives the person the chance to understand the physical objects. The subsection D is Imagination, *eikasia*, which is used to know the images. As Foley explains, he has preferred to “follow one general tendency in the literature of labeling the section representing the Forms with the letter ‘A’ and treating it as the longest subsegment because Forms are first in order of importance”(Foley 2008, footnote 1, p. 1).

The different length of the subsections of the line segment is traditionally used to represent the different cognitive importance of the objects which correspond to them and of the intellectual faculties necessary to understand these objects. Longer subsections represent objects more difficult to grasp and more advanced cognitive faculties, necessary to investigate these objects. Foley comments the lines of the *Republic* quoted in 1., in which Plato exhorts his readers to investigate further the subject of human cognitive progress, stating that even if it seems that the Platonic indications to divide the line segment entail the existence of two middle subsections of *equal* length, when we analyze further this schematization we see that “the two middle subsegments are *unequal* because they represent mental states of unequal clarity, and possibly also objects with unequal degrees of reality”(Foley 2008, 1).

I disagree with Foley because I think that the words of Plato cited above have not to be interpreted only within the cognitive space of the four sectors of the line segment that we have examined. On the contrary, these sectors are the starting point of an intellectual progress which is not described in the dialogues but is

originated by them. Plato's words, in my interpretation, are an exhortation to keep in mind that the content of the dialogues is just one chapter of the Platonic book of knowledge.<sup>2</sup> This must guide our interaction with the Platonic text, in case we decide to cooperate with it, as I have done, accepting to stay within the conceptual boundaries given by the Platonic conception of knowledge, which culminates with the apprehension of the purely intelligible. In my reconstruction of what the Platonic account of human intellectual progress could be, I am aware of the role of his written words, in respect to the larger cognitive project that the philosopher indicates. But I am also aware that this broader theoretical framework, even though it has to respond to the Platonic idea of truth, which has to be totally separated from the empirical, leaves us the necessary intellectual space to shape this truth with our contributions.

This positive characteristic of Platonic philosophy leads to the fact that my reconstruction of the stages of human intellectual development,<sup>3</sup> respects and is guided by the Platonic principles about knowledge and truth but it is disputable because it cannot respond to a precise Platonic description. Nevertheless, I need to make an assumption in order to progress with my research on Plato's ideas about human rational growth. I take on board a piece of scientific method to elaborate my theory about what could be the stages of cognitive progress, which should be added to those described in the *Republic*. In science, when there are testable elements which present variations which are not in line with what was theorized about their properties, it is possible, before rejecting the theories about those elements, to hypothesize that the unpredictable variations are generated by other elements, whose existence was not taken into consideration before.

This is the way in which in the nineteenth century the planet Neptune was discovered: the motion of Uranus was considerably different from that predicted through the Newtonian gravitational theory. In order to find a solution to this problem it was hypothesized that there should be a previously undetected planet close to Uranus. The attraction between this hypothetical planet and Uranus had to be considered the cause for the departure of Uranus from its initially predicted orbit. Once this hypothesis was assumed to be true, it was possible to test its content, checking with a telescope for the presence of an undiscovered planet. This led to the first sighting of Neptune, saving Newton's gravitational theory (Chalmers 1976, 78).

In our case, the Platonic excerpt which we have taken into consideration via Foley's comment, is the unpredictable effect which confirms our theory about the existence of stages of cognitive development, which add subsections to the line segment used by Plato to represent human intellectual progress. These subsections are indicated with A', B', C', D' in the schematization below and they are our Neptune, which has not been noticed before.



As we have seen, Foley has chosen to represent with A the Forms, pointing at the significance of this object and of the cognitive capacity correspondent to its understanding, through the use of a subsection of the line segment of intellectual progress larger than the others. In my line segment, the subsection A represents the images and the cognitive capacity necessary to grasp them. When we are able to understand D, the Forms, we reach a superior level of intellectual development. Starting from

this epistemic moment, we are able to begin the investigation of the purely intelligible, which is for Plato the highest rational achievement.<sup>4</sup> In the dialogues, there is no indication of how this investigation can take place. I have hypothesized that there can be stages of rational progress also in the cognitive development of the individuals who are already able to investigate the purely intelligible. For this reason, I have also hypothesized that the analysis of the purely intelligible has to begin with an empirical aid, as it happens in the first stages of rational development described by Plato. These stages are represented by the subsections A and B of my line segment, that are, as A' and B', still related to the empirical. With this notation, I suggest the correspondence between the stages of cognitive development, A-D, necessary to reach the epistemic point in which we are able to start the investigation of the purely intelligible and the stages of cognitive advancement, A'-D', of the individuals who are already able to research the purely intelligible.

In order to stress that the description of human intellectual evolution given in the *Republic* is only the first part of the cognitive progress of the individual, I have chosen to call the four sectors of the line segment traced in the *Republic*, theoretical childhood (A-D); the extension of this line segment is theoretical adulthood (A'-D'). I am using the term *theoretical* having in mind the relation between *theōreō* and *oraō*, which implies a *process of cognition which starts with the vision, instantiated through physical or intellectual eyes*. Thus, theoretical childhood will be that stage of cognition in which the speculations are in their childhood because the intellectual eyes are not yet looking in the right direction. With the expressions theoretical childhood and theoretical children, I am *not* referring to *real* children and their cognitive development but I am defining phases of rational evolution, one

intellectually more advanced than the other, coherent with Plato's indications.

Plato states explicitly what are the objects analyzed during the rational progression from A to D. The purely intelligible is the most complex object that the human reason can examine. Thus, it is plausible that its knowledge takes place in stages and that the beginning of the investigation of the purely intelligible is still informed by the tangible, as means to reach the purely intelligible. We do not know whether A', B', C', D' correspond to different objects which reveal different aspects of the purely intelligible or whether different cognitive layers of the purely intelligible are the objects of investigation in A', B', C', D'. But my addition of subsections in the line segment of cognitive progress described by Plato has not the purpose of providing the final answer about the Platonic account of human intellectual development.<sup>5</sup> My representation of this account wants to emphasize that the individual rational growth, as envisaged by Plato, does not end in the description of the *Republic* (R. VI 509d-510) but it continues with stages of rational development complementary to those traced by the Platonic words. This extension, grounded in the lines of the *Republic* commented by Foley, (R. VII534 a) is my way of responding to the request for collaboration with the text which is, as we have seen, a fundamental aspect of the Platonic dialogues.

In my representation, all the sectors of the line segment have equal length.<sup>6</sup> This does not mean that I think that there is no theoretical difference among the objects and mental stages which correspond to the parts of the line segment. In fact, the text of the *Republic* provides fuel for discussion of equal or unequal length of the subsections of the line segment. Joining this discussion would serve no purpose in my interest on this representation of the

Platonic account of intellectual development. This interest is focused on the *equal* epistemic significance that *each* subsection has for the individual rational development. Maintaining the focus on the *function* of each epistemic stage of the line segment is crucial to grasp the significance of this representation for the understanding of the nature and potentiality of human rationality according to Plato.

We have seen so far the phases of development of theoretical childhood and adulthood. Now I want to present the basics<sup>7</sup> of my reconstruction of what could be for Plato the different means, or techniques as I call them, that favour rational progress in each of the phases which are part of theoretical childhood and theoretical adulthood. The technique that Plato has chosen to make theoretical children evolve cognitively is the use of natural language. The beginning of this analysis is given by the quotation from the *Phaedrus* that we have already taken into consideration in 1. In that excerpt, Socrates reports a dialogue between Thamus and Theuth about the art of writing. For the present purposes, our attention has to be focused on the distinction, made in that excerpt, between knowledge which stems from *external reminders* and *knowledge which emerges exclusively from the reasoning capabilities of the individual*. In 1., quoting the *Meno*, we have spoken about the ability of the *reason* to remember, to *recollect*, originating knowledge by itself. But when we have not yet developed this skill we need the words, external reminders of our cognitive potentialities.

We need only to be *reminded* about our intellectual capacities because even during a phase, theoretical childhood, in which we have not yet reached a high degree of intellectual sophistication, we already possess the skills to attain this goal. This is stressed by Plato in the following lines:



Education isn't what some people declare it to be, namely, putting knowledge into souls that lack it, like putting sight into blind eyes....*the power to learn is present in everyone's soul...*education...it isn't the craft of putting *sight into the soul*. Education takes for granted that sight is there but that it isn't turned the right way...and it tries to redirect it appropriately. (R. VII 518c-d. My emphasis.)

These words are part of Book VII of the *Republic*, where the allegory of the cave shows the necessity that the eyes who have always lived in the obscurity of the appearance of knowledge adjust gradually to the sight of its bright reality. This excerpt points to the *graduality* of the process of human intellectual development, as it is confirmed from the context in which these lines appear. The reasoning ability is a skill proper of the human beings and it belongs to everyone of them. Nonetheless, to make sure that the cognitive eyes look at the truth, it is necessary that they are appropriately stimulated. This will avoid the danger emphasized by Mattéi in the lines quoted in 1.: people stop at the spectacle created by Plato's *words* without investigating its *function*.

We have taken into consideration words to point at their usefulness for the rational growth of theoretical children. Nevertheless, natural language is not the appropriate technique for the rational stimulation of theoretical adults. As we have seen, the object of investigation of theoretical adults is the purely intelligible. To understand what could be an adequate technique to promote the development of this higher-level thinking, I am going to start from Foley's emphasis on the importance attributed by Plato to mathematics. As we have seen in Foley's discussion of his rendition of Plato's line segment which represents objects and the

cognitive faculties necessary to understand them, mathematical objects are the first point of entrance in the realm of the intelligible. This is what Foley explains, emphasizing

the tremendous importance that mathematics has in Plato's account of philosophical development. The study of mathematics serves as a bridge between physical objects and the Forms. Learning to think mathematically is presented as a necessary condition for thinking philosophically because mathematics is what leads us from concern for physical objects to understanding of eternal objects. Once this transition to eternal objects has been made, it is easier to study the Forms. (Foley 2008, 12)

We have stressed the significance of Foley's thought about the Platonic text as stimulation for a research which has not to end with those written words. Now he points at the need of considering the crucial role that mathematics plays in Plato's philosophy, as the bridge between an inferior level of rational development, which can know only via the physical realm, and a superior intellectual refinement, which is able to grasp the non-sensible, the Forms.

I agree with Foley's statements about the significant role that mathematics plays to reach the highest intellectual goal according to Plato, the knowledge of the purely intelligible. Nevertheless, Heath stresses a difference between mathematical and dialectical method in Plato which can make us think that mathematics is imperfect in comparison with dialectic and it cannot be the technique which promotes a higher-order development of human rationality:

Plato distinguishes two processes: both begin from hypotheses. *The one method*

cannot get above these hypotheses but, treating them as if they were the first principles, builds upon them and, with the aid of diagrams or images, arrives at conclusions: *this is the method of geometry and mathematics in general*. The other method treats the hypotheses as being really hypotheses and nothing more, but uses them as stepping-stones for mounting higher and higher until the principle of all things is reached, a principle about which there is nothing hypothetical; when this is reached, it is possible to *descend again*, by steps each connected with the preceding step, to the conclusion, a process which has no need of any sensible images but deals in ideals only and ends in them; this method, which rises above and puts an end to hypotheses, and reaches the first principle in this way is the *dialectical method* (Heath 1921, 290. My emphasis).

These lines should not be considered as the base for an exclusion of mathematics from the realm of theoretical adulthood. This would be an incorrect inference which can be avoided if we take into consideration the different levels of mathematical complexity.

The first level of mathematical complexity can be associated with an axiomatic approach which can be defined as top-down axiomatic approach. This is “the method of geometry and mathematics in general.” (Heath 1921, 290) it helps us to prove that results are correct (Greenberg 1974, 8) using the axioms, which are never questioned, and the logical consequences we derive from them. With this method results are logically deduced from unquestioned axioms, which are the foundations which ground the mathematical structure.<sup>8</sup> Greenberg explains to us what an axiom is, emphasizing that

If I wish to persuade you by *pure reasoning* to believe some statement S1, I could show you how this statement follows logically from some other statement S2 that you may already accept. However, if you don't believe S2, I would have to show you how S2 follows logically from some other statement S3. I might have to repeat this procedure several times until I reach some statement that you already accept, one I do not need to justify. That statement plays the role of an *axiom* (or *postulate*). If I cannot reach a statement that you will accept as the basis of my argument, I will be caught in an “infinite regress,” giving one demonstration after another without end. (Greenberg 1974, 9)

Greenberg's words point to the fact that the *axioms* are grasped through *pure reasoning*; thus, they lead us directly towards the *purely intelligible*. This reminds us of the role of mathematics in the redirection of our cognitive sight towards the intelligible, which Foley was emphasizing.

I have pointed at the existence of two levels of mathematical complexity. We have seen briefly the utility of the geometrical axioms to move from the tangible to the intelligible. This focus on the intelligible is for Plato fundamental to evolve intellectually *till to the point in which we become theoretical adults*. *The mathematics utilized by theoretical adults*, already emerges from Heath's words about the dialectical method. When mathematics is applied to the understanding of complex problems, it is not anymore based upon axioms, which do not require any reconsideration. On the contrary, at this level of sophistication, the consequences of the problem have to be utilized to reconsider the truth of the premises. (Russell 1973, 273-274) In this case, we have not a rational movement which merely

goes from an element to its mathematical consideration via a mathematical principle which will not require any reevaluation. This is the way in which the axiomatic approach which we defined as top-down works and its relative simplicity allows its utilization by theoretical children, favouring their cognitive progress towards theoretical adulthood. But, as Foley has highlighted, for Plato the highest point of intellectual evolution is reached when the purely intelligible is the only subject of investigation. At that speculative level, theoretical adults have to try to solve problems whose complexity demands to go back from what has been considered a correct result, a correct consequence of their thinking, to its premise. This axiomatic approach can be called bottom-up since the progress of theoretical adults in the understanding of the consequences of their line of reasoning will illuminate the comprehension of the related premises. We are going to know more about this last kind of axiomatic approach, analyzing it in connection with visual thinking.

I have pointed to the fact that the written words are useful reminders for individuals whose intellectual skills have not yet been totally developed. When Plato's readers reach the cognitive complexity of theoretical adults they have no necessity of the mediation of a written text to progress intellectually. Indeed, this text would be very difficult to compose because it should describe the myriads of intellectual routes which can be chosen by a mind whose capacity of selection is not restrained by cognitive mistakes. This kind of description would be not only very challenging to write but also useless since the only people who could grasp its content would be those who have already reached a level of intellectual maturity which renders the written reminders pointless. This level of development of the human intellectual capacities is not the object of a *direct* Platonic description. Thus, my

reconstruction of theoretical adulthood is, in a sense, solidly grounded in Plato's text because it is a reconstruction of a phase of human rational development based, as we have seen, on the effects that this cognitive phase, theoretical adulthood, provokes on another phase, theoretical childhood, directly described by Plato. Nevertheless, the ground of theoretical adulthood is *meant* to be shaken by the contributions of minds which have no fear to leave the place of tradition to develop innovative researches. Consequently, I am ready to admit not only that my idea of theoretical adulthood can be criticizable but also that if it was not criticizable, it would not be that territory of novelty, correspondent to the Platonic choice of leaving this cognitive zone to the rational talent of his readers.

## 2. THE MENO AND VISUAL THINKING

We have seen that the criticisms of Plato's words are not mere accidents: their occurrence is provoked by the dialogical interaction to make them become part of the philosophical message itself. As we said, this rational stimulation is not meant to make us accept Plato's idea of truth. We, as readers of the Platonic dialogues, are rationally stimulated by Plato to discover a rational sophistication of which we were not aware. We are guided by someone who knows more than we do, but we are guided by him through a dialogical exchange. This method makes us discover the rational resources which give us the chance to critically evaluate the thoughts of the person who is intellectually guiding us, acquiring at the same time the capability of completing his own system and the independence from its content. Through the dialogues, Plato is inviting us to reflect on *our* cognitive resources to develop them autonomously.

An example of the importance of the investigative freedom of the rational creature is found in the dialogue *Meno* where Meno's slave will discover that he possesses the intellectual ability to find an answer to a geometrical problem thanks to the dialogical interaction with Socrates. The slave is not pressured to accept the point of view of an earlier theorist or Socrates' beliefs; indeed, Socrates never expresses his point of view but he questions his interlocutor to develop in him the awareness of his intellectual abilities. The cognitive growth of Meno's slave takes place in the fictional stage of the *Meno*: the slave's answers are decided by Plato as part of his fictional creation but this creation points at the importance of the independent rational activity of the subject of a dialogical interaction. Even when the contribution of Meno's slave is limited to an affirmative or negative answer his replies reveal his own rational activity, stimulated by the words of his interlocutor but developed independently from them (see in particular *Men.* 81c-e). In fact, the solution of a geometrical problem by someone who has never studied geometry requires a reasoning which, even if it is not fully recorded in the dialogue, is present in the correctness of the slave's answer. Thus, the slave's answers are not perfunctory because they are signalling a process of active reflection, required to reply correctly to the questions presented. In the *Meno* the slave is not questioned to learn Socrates' truth, he is questioned to discover that there is truth in himself.

The slave in the *Meno*, through Socrates' questioning, acquires conscience of his rational abilities but what kind of thinking is involved in the reasoning of the slave who gradually realizes to possess the cognitive capacity to know a geometrical truth? An answer to this question comes from Marcus Giaquinto's research. He

has worked on the epistemological importance of visual thinking in mathematics.<sup>9</sup> According to Giaquinto "the oldest and best known discussion of visual *discovery* is to be found in Plato's *Meno* (82b-86b)" (Giaquinto 2008, 32. My emphasis). Giaquinto explains that it is usually considered impossible to *discover* a geometrical theorem thanks to visualization. This happens because, when visualizing and seeing are compared, it is usually felt that visualizing is no better than seeing (Giaquinto 2007, 67). This is due to a misleading comparison: in fact, "while the *experience* of visualizing is similar to the experience of seeing, the *epistemic role* of visualizing can be utterly different from the primary, evidence-providing role of seeing...So the fundamental mistake here is to assume that the epistemic role of visual experience, whether of sight or imagination, must be to provide evidence. In view of its non-evidential role we can say that visualizing...is part of an *a priori* means of...discovery" (Giaquinto 2007, 67). Visual discovery, for Giaquinto, is an *a priori* and "it consists in the operation of a synthesis of visually triggered belief-forming dispositions. Hence it may be appropriately regarded as a synthetic *a priori* route to knowledge" (Giaquinto 2007, 67-68).

To understand better the *epistemic* role of visualization according to Giaquinto, it is necessary to come back to the *Meno*. There (*Men.* 81e-86c), Plato famously presented a visual way of discovering a simple fact of geometry: if a diagonal of one square is a side of another square, this other square has twice the area of the first (Giaquinto 2007, 12). Giaquinto emphasizes the necessity that every geometrical discovery has a starting point. Thus, the initial challenge of this Platonic visual discovery is this: "how can we acquire *basic* geometrical knowledge?" (Giaquinto 2007, 12). According to Giaquinto

In having geometrical concepts for shapes, we have certain general *belief-forming dispositions*. These dispositions can be triggered by experiences of seeing or visual imagining, and when that happens we acquire geometrical beliefs. The beliefs acquired in this way constitute knowledge, in fact synthetic *a priori* knowledge, provided that the belief-forming dispositions are reliable (Giaquinto 2007, 12. My emphasis)

In this excerpt Giaquinto explains that a visual discovery involves the activation of dispositions, that he defines as “belief-forming dispositions” (Giaquinto 2007, 12) that come with possession of certain geometrical concepts (e.g. square, diagonal). What *triggers the activation* of these dispositions is conscious *visual experience*. A belief acquired in this way is *non-empirical*, “because *the role of experience is not to provide evidence*. At the same time, some *visual experience is essential for activating the relevant belief-forming disposition*” (Giaquinto 2007, 47. My emphasis). Giaquinto notices that in some cases, as in the case of the *Meno*, the mode of belief-acquisition is fast, thus the resulting belief seems to the subject immediate and obvious (Giaquinto 2008, 33). In very many cases we are unaware of the cause and occasion of the acquisition of a belief. In fact,

having a belief is not a manifest state like a pain state—some of our beliefs we are unaware of having—and the transition from lacking a certain belief to having it may also occur without awareness...One may not get a firm belief all at once; to acquire a firm belief by activation of a belief-forming disposition, activations on several occasions may be needed. *But the point is unchanged: there is no anomaly*

*in the fact that we are usually unaware of those occasions.* (Giaquinto 2007, 39. My emphasis)

In the case of the *Meno*, one gets the belief almost immediately, that is, “without any subjectively noticeable period between visualizing and getting the belief. *Immediacy* suggests that to explain why *visualizing* leads to the *belief* we should look to the *visualizer’s prior cognitive state*. One hypothesis is that the subject’s prior cognitive state included tacitly believing B. *This kind of view was proposed by Plato. On Plato’s view the experience of visualizing triggers retrieval of the tacit belief B*” (Giaquinto 2007, 60. My emphasis).

Giaquinto’s research has helped us to see in the *Meno* an example of visual *discovery*. Visual thinking is based upon visual activation of belief-forming dispositions. Thanks to these belief-forming dispositions we acquire concepts, such as that of square or diagonal, which allow us to discover, as in the case of the slave in the *Meno*, geometrical truths. In the case of the *Meno* visualization triggers *immediately* the relevant belief-forming dispositions. This entails that the subject’s prior cognitive state already included these beliefs. This is in line with what Plato states about recollection in the *Meno*: in 1. we have seen that, according to Plato, the individuals possess wisdom within themselves. For the philosopher the process of learning is memory, *recollection*, of what is *inside* us.

For Giaquinto visualization has *epistemic* importance since its role is *not* that of providing evidence; rather, visualization, activates the relevant belief-forming dispositions which render possible a visual discovery. This epistemological role of visual thinking has contributed to make us realize *how* the Platonic dialogues can stimulate cognitively the readers; in fact, the words of the *Meno* promote in Plato’s readers an *episte-*

mic progress via *visual thinking*. Moreover, we have seen that visual discovery, as the geometric discovery of the slave in the *Meno*, is a synthetic *a priori* discovery, in which visually triggered belief-forming dispositions are synthesized. This character of visual thinking relates the discovery of the slave in the *Meno* with what, according to Plato, is the best use of geometry. In fact, the philosopher, in the seventh book of the *Republic*, states that “if geometry compels the soul to study *being* it’s appropriate, but if it compels it to study *becoming*, it’s inappropriate” (*R.* VII 526e. My emphasis).

### 3. VISUAL DISCOVERY AND MENTAL NUMBER LINES

Giaquinto observes that exists an innate propensity to represent ordered systems of items, such as alphabets or months, as a line. Our disposition to form a mental number line representation once we have acquired a written numeral system may be a special case of this propensity (Giaquinto 2007, chapter 6. See in particular pp. 99; 116). We typically visualize a number line as a graphical line with numbers represented as positions on the line ordered from left to right for individuals in Western cultures (Giaquinto 2007, 107). There are many possible variations. “What seems likely to be constant is that each number is represented by a position on the line (or in the row of numerals) relative to a unique *origin*,....and that the *size of the number is represented by the relative distance between the origin and the number position*”(Giaquinto 2007, 108. My emphasis).

*Visual number lines* are important in our *mathematical thinking* because the *visual argument* is persuasive and makes the correctness of the proposition obvious in a *direct* way: “whole number addition can be represented easily as a

movement to the right from the position marking one addend by the length representing the other addend, the result being represented by the end position (or the length of the segment from the origin to the position). Whole number subtraction  $n-k$  can be represented as a leftward movement from the position representing  $n$  by the length representing  $k$ , the result being represented by the end position.... we also have representations of multiplication, division, and rational numbers in terms of the number line...” (Giaquinto 2007, 111). The *epistemic* result is achieved by deploying one’s implicit grasp of these facts of representation together with vision or visual imagination and some simple deduction (Giaquinto 2007, 115). The disposition to integrate symbolic and diagrammatic representations is found in innovative mathematicians and its fruitfulness is beyond dispute (Giaquinto 2007, 116).

In the first quotation from Plato cited in this work, the philosopher is summarizing his idea of cognitive progress, and he mentions a ratio pertinent to the sectors of the line segment that has been utilized to render the different stages of rational advancement. In those lines, as we have seen through Foley’s comment, Plato exhorts his readers to transcend the text. Plato asks his readers to collaborate with the text. Giaquinto has just given us an idea of one possible way to respond to this request. In fact, we could reflect on the reasons why Plato has chosen to represent phases of mental growth via a schematization which is directly related to an innate propensity of the human beings to represent *ordered* systems of items, as a line. The most immediate thought is that Plato considered the stages of mental growth as the items that he was presenting in an order. Nevertheless, Plato does write that there is much more to know about cognitive growth. Moreover, he does point to the concept of *proportion* among

the phases of epistemic advancement. As Giaquinto has helped us to observe, human beings have an innate propensity to order a numeral system via a mental number line; thus, there are reasons to think that Plato has chosen to represent the phases of cognitive progress via a line segment to move the reader's attention towards the importance of *mathematics*. Assuming that this could be true, new questions arise: *where* should mathematics exercise its power? Within the four subsegments of cognitive progress mentioned by Plato or there could be a more advanced phase of intellectual development where mathematics can express its full potential? It is not difficult to notice that these reflections could be available to Plato's contemporaries and they have not to be necessarily circumscribed to the modern reader.

As we have seen (1.1.1.), Foley has answered to the Platonic request of collaboration with his text, working on the length of the subsections of the line segment that represents for Plato cognitive progress. This hermeneutic approach entails a reading of the schematization of the line as a mere *diagrammatic* representation. There is no reflection on the *symbolic* meaning of the diagram itself. Recall the distinction introduced by Giaquinto about visualization: it is essential to recognize the difference between the *experience* of visualization and its *epistemic* role if we want to *discover* via visual thinking. If we limit our reflection on the line segment introduced by Plato to represent his idea of cognitive progress to the measurement of the length of the subsections which compose it, visualization has no epistemic role and no visual discovery can originate from it.

An objection to this line of reasoning could be based on the fact that when Foley analyzes the length of the sectors of the line segment, he specifies that their different lengths represent a possible different equality of the cognitive

faculties and of the objects of cognition correspondent to the sectors themselves. Nevertheless, these considerations are applied to specific aspects of a schematization with *no* reflection on *the reasons why* Plato chose *that* schematization to represent intellectual progress. As Giaquinto has helped us to realize, the schematization of the line segment is mathematically tainted. Thus, the lack of reflection on the reasons why this specific diagram has been chosen by Plato to represent cognitive progress, it is equivalent to a use of mathematics that Plato criticizes. In fact, for Plato, mathematics has not to be used as retailers and tradesmen do, just to be able to buy and sell, but it must be used to turn the soul upward, compelling it to discuss the *nature of the numbers* and in this way moving from *becoming* to truth and being (R. VII 525b-c. My emphasis). Thus, observing a diagram which is used to *order* systems of items, such as the numeral system, we should not limit ourselves to the *experience* of visualizing but we should be able to leave the *empirical*, the realm of *becoming*, to turn our attention to the *truth* of the *epistemic* role of visualization.

### 3.1 MENTAL NUMBER LINES AND INFINITE STRUCTURES

Giaquinto takes into consideration a particular case of mental number line: he analyzes the mental number line which allows us by means of visual representation to know an infinite structure, the structure of the natural numbers. Giaquinto, working on *visual cognition of an infinite structure*, refers to the "structure of the finite cardinals under their natural 'less than' ordering. This structure, which I will call 'N', is shared by the set of arabic numerals of the decimal place system in their standard ordering" (Giaquinto 2007, 226; See also Giaquinto 2008, 53).

As Giaquinto notices, “an obvious problem with the idea that a mental number line provides a grasp of the *natural number structure* is that we cannot see or visualize more than a finite part of any such line. When it comes to actual images (or percepts) something like Fig. 2.6 will be the best we can do” (Giaquinto 2008, 53. My emphasis).<sup>10</sup>



Fig. 2.6

The fact that we cannot see or visualize more than a finite fragment of any instance of an infinite structure is not an insurmountable obstacle. For Giaquinto there are two kinds of visual representations, visual category specification and visual image. “A *visual category specification* is a set of related feature descriptions stored more or less permanently; a *visual image* is a fleeting pattern of activity in a specialized visual buffer, produced by activation of a stored category specification. What is impossible is an infinitely extended visual image. But it is possible, and not at all puzzling, that a category specification specifies a line with no right end, one that continues rightward endlessly” (Giaquinto 2007, 227. See also Giaquinto 2008, 54).

In having a visual category specification for the mental number line, “we have a grasp of a *type of structured set*, namely a set of number marks on a line endless to the right taken in their left-to-right order of precedence. Secondly, we can have knowledge of the *structure N* as the structure of a ‘number line’ of this type” (Giaquinto 2008, 56. My emphasis. See also Giaquinto 2007, 228). Giaquinto has emphasized the importance of mental number lines for the cognition of some infinite *structures*; in particular, he has taken into consideration

the natural number *structure*. We are going to see what *structuralism* is and what could be its relation to Plato’s philosophy.

### 3.1.1 STRUCTURALISM AND PLATO

As we have just seen, Giaquinto shows us how the infinity of the natural number structure can be rendered via a mental number line with no right end. This representation abstracts away from the nature of the objects, the natural numbers, which instantiate the natural number structure. In fact, according to *structuralism*, *numbers*, e.g., in the natural number structure, should be treated as *positions in structures*. For the structuralist, “mathematics is seen as the investigation...of ‘abstract structures’, systems of objects fulfilling certain structural relations among themselves and in relation to other systems, without regard to the particular nature of the objects themselves....the ‘objects’ involved serve only to mark ‘positions’ in a relational system; and the ‘axioms’ governing these objects are thought of, *not* as *asserting definite truths*, but as *defining* a type of structure of mathematical interest” (Hellman 2005, 536-537). We will come back to Hellman’s words shortly. Now, I want to take into consideration a particular instance of structuralism, Shapiro’s *ante rem structuralism*. The basics of this kind of structuralism are well explained by Sereni:

Arithmetic assertions...are not centred on particular objects...Rather, they are based upon the *positions* of the progression structure. For example, the assertion ‘ $3 < 5$ ’ does not state that a particular object, 3, is in the relation ‘being minor of’ with another particular object, 5. Rather, it states that the *position* of the progression



structure that we call ‘3’ (that will be the third or fourth *position* of the structure, according to the fact that we choose to make the structure begin with 1 or 0) comes before, according to the order relation that exists among these *positions*, the *position* of that same structure that we call ‘5’. *The fact that exist particular objects, numbers, or other abstract objects, or concrete objects, that occupy those positions and that constitute a system that exemplifies the structure in question, is something that lies outside the object of arithmetic and the significance of its assertions. There could exist natural numbers, occupying the positions that we call with their names;... or there could exist nothing that satisfies the relations of the progression structure. Independently from this, the object of arithmetic-that specific structure- does not change, and its theorems remain true descriptions of that object.* (Sereni 2020, 166-167. My translation. My emphasis)

These words have helped us to understand what *ante rem* structuralism is: it is a kind of structuralism that *ignores the individual properties of the objects*, that are *irrelevant*, and it considers only an object as a *position* in a structure.

Shapiro states that *ante rem* structuralism is an instance of the view that he calls ‘realism-in-ontology’ (Shapiro 2006, 142). He also points to the fact that “*ante rem* structuralism is a variant of traditional Platonism” (Shapiro 2011, 130. See also Shapiro 2006, 142). In Shapiro’s structuralism there is an “*existential* commitment to both *structural universals* and their *positions*. The structural universals so described are ‘*ante rem*’ because, *like Plato’s Forms, they exist independently of the systems that exemplify them*” (MacBride 2008, 156. My emphasis). The “*ante rem* structuralist takes a Platonic view

of structures: they exist and are available for mathematical description as complex objects in their own right, *whether or not exemplified by any independent collection of objects*” (Wright 2000, 330. My emphasis).

Shapiro connects *ante rem* structuralism with Plato’s philosophy: for Plato reality and truth are disentangled from the *empirical* realm and can be found in the *purely intelligible*, in the same way, for Shapiro, it is irrelevant the *empirical* existence of objects that exemplify the structures that he is taking into consideration; these objects exist *ontologically*, as those *positions* in a structure which can be grasped via an act of *intellection*. *Both for Shapiro and for Plato, the truth is not in the empirical but in the intelligible dimension.* The existence of the structures is posited by Shapiro via an axiomatic theory of structures. Shapiro’s structures are axiomatically characterized (Sereni 2019, 253); nevertheless, Hellman has clarified that the axioms, governing the objects that in structuralism are positions in a structure, do *not* assert *definite* truths but they *define* a kind of structure of mathematical interest (Hellman 2005, 537). The axiomatic approach connected to structuralism can be thus related to the axiomatic approach that in **I.1.1.** has been called as bottom-up: there are not axioms, which are never questioned, used to logically derive mathematical truths from them; on the contrary, there are axioms whose truth can be reconsidered in light of the results of the mathematical problem examined. This is an axiomatic approach proper of a higher-level of mathematical complexity, appropriate to the investigations of theoretical adults who, as we have seen, analyze the purely intelligible. Recall, we have distinguished between two levels of mathematical complexity, the first level, “the method of geometry and mathematics in general” (Heath 1921, 290), was associated with an axiomatic approach that we defined as top-

-down axiomatic approach: with this method, results are logically deduced from unquestioned axioms. This level of mathematical complexity is useful to turn our rational attention from the tangible to the intelligible. This focus on the intelligible is for Plato fundamental to evolve intellectually till to the point in which we become theoretical adults. The mathematics utilized by theoretical adults is based on a bottom-up axiomatic approach. At this level of sophistication, the consequences of the problem have to be utilized to reconsider the truth of the premises.

I have associated the investigation of the purely intelligible proper to theoretical adults with the level of mathematical complexity of structuralism.<sup>11</sup> It can be objected the existence of theoretical adulthood. I have never stated that the phase of superior cognitive development that I label as theoretical adulthood is the only way to respond to the cognitive stimulation of Plato's text. This would be contrary to the non-indoctrinative Platonic higher-order pedagogy which, as we have seen, presents to the reader what Plato's idea of truth is, but it does not impose the acceptance of this truth. According to my hermeneutic approach, the words of Plato's dialogues are meant to stimulate cognitively the readers. In this way, they acquire conscience of their intellectual capacities. The exercise of these cognitive skills can result in a radical criticism of Plato's idea of truth. I have accepted this idea and I have responded to the Platonic request of collaboration with his text, elaborating a new theoretical framework, characterized by two moments of epistemic growth, theoretical childhood, which corresponds to the description of cognitive development provided by Plato in the *Republic*, and theoretical adulthood, which is not the object of a direct Platonic description.

As I have clarified in 1.1.1., I have used the term *theoretical* having in mind the relation be-

tween *theōreō* and *oraō*, which implies a *process of cognition which starts with the vision, instantiated through physical or intellectual eyes*. As we have seen, Plato in the *Republic* (R. VI 509d-513e) chooses to convey his idea of intellectual development utilizing the schematization of the line segment. As Giaquinto has helped us to notice, visualizing has an *epistemic* role and it contributes to visual discovery. In particular, Giaquinto has pointed to the importance of mental number *lines* for visual discovery and he has showed how mental number lines can make us grasp what an infinite structure is. Giaquinto has acknowledged the relation between Plato's philosophy and visual cognition, taking into consideration the geometrical discovery of Meno's slave in the *Meno*. I have reflected *on the reasons why* Plato could have chosen the schematization of the line segment to render his idea of cognitive progress. He proposed a schematization which could immediately engage the reader; nevertheless, in my opinion, Plato's goal was not that his readers stopped at the mere *empirical* visualization. Plato's readers had to question the choice of the philosopher, they had to reflect on the *epistemic role* of visualization.

As I said in 3., this reflection can be stimulated by the Platonic invitation to consider the *proportion* among the phases of epistemic advancement that have been presented (R. VII 534a). In this way, Plato's readers can start a line of reasoning centred on the fact that the schematization of the line is mathematically tainted. This reasoning can move on considering the possibility that a line segment is used to render an infinite structure; Plato's readers can ponder on the nature of an infinite structure and the appropriate means to know it. A possible outcome of this line of reasoning is the realization that an infinite structure can be the infinite structure of the natural numbers. Thus, mathematics can make us know it. The

mathematics of the infinite cannot be chained to the empirical so Plato's readers can start to think about a kind of mathematics which abstracts away from the properties of the natural numbers. Plato's readers, as finite human beings who investigate the infinite, can also start to reflect upon their cognitive limits. We are going to take into consideration this problem in the next section of this work.

### 3.1.1.1 EPISTEMOLOGY IN ANTE REM STRUCTURALISM: THE ACCESS PROBLEM

As we have just seen, *ante rem* structuralism is a theory about what (mathematical) *universals* there are. Shapiro offers a stratified epistemology,<sup>12</sup> in which each stage corresponds to the acquisition of knowledge of successively more complex mathematical structures. Knowledge of structures begins with our capacity to recognize small, finite, instantiated patterns or structures; for example, short strings of numerals. The subject observes one or more systems of objects arranged in various ways and she *abstracts* away from the irrelevant *tokens*, apprehending the *types (universals)* under which they fall. This abstractionist step of Shapiro's epistemology allows the individuals to know small cardinal number structures but since our powers of perceptual discrimination are essentially limited, our ability to abstract types from tokens with which we are acquainted will not provide us with knowledge of large natural numbers structures such as the 1000 pattern. Thus Shapiro postulates the existence of a faculty of *projection*: this faculty enables us to arrange the patterns obtained by simple abstraction and *recognize* that they themselves exhibit an overarching pattern. This yields knowledge of large finite structures, and eventually know-

ledge of the natural number structure itself. But the faculty of projection is still too limited for mathematical purposes. To deal with still larger structures an alternative epistemological strategy is proposed: Shapiro poses the need of a formal language that provides appropriate definitions of the structures to allow us to know them. It is consequently our ability to grasp *direct descriptions* of large infinite structures that grounds our knowledge of them.

These steps of Shapiro's epistemology, according to MacBride, do not provide any answer to the problem that he defines as "the access problem" (MacBride 2008): how can mathematicians reliably access truths about an *abstract* realm to which they cannot travel and from which they receive no signals? (MacBride 2008, 155. My emphasis). For MacBride the problem consists in a tension between Shapiro's realism in ontology and naturalized epistemology: how can a physical being located in a physical universe know the abstract realm, which includes *ante rem* universals and infinite structures (MacBride 2008)? Shapiro's reply (Shapiro 2011, 149. My emphasis) to MacBride's doubts is that

My game, again, is to provide a justification for a philosophical interpretation of mathematics, an interpretation which includes a thesis concerning what mathematics is about-*ante rem* structures. This philosophical interpretation is not a deductive enterprise, where I would have to start with non-mathematical, self-evident premises. *This is a different game from showing a sceptic that mathematics itself is true and known.*

According to Shapiro, the goal of his research is to demonstrate that mathematical knowledge *just* is knowledge of *ante rem* structures. This has not to be proved from accepted non-mathematical premises. Shapiro's research

aims at studying *ante rem* structures. As we have seen, these structures possess an *ontological* reality independent from the empirical existence of entities which physically instantiate them. This focus on the universal rather than the empirical realm is common to Shapiro and Plato, as Shapiro himself acknowledges (Shapiro 2006, 142; Shapiro 2011, 130). Both Shapiro and Plato do not tell us where their universal evidence comes from. But Plato has chosen to provide us with cognitive stimulations which give us the chance to criticize his system and every aspect which characterizes it. In this research we have taken into consideration how Plato's text can stimulate us cognitively via visualization, realizing the *epistemological* importance of visualizing: "Some 'pictures' are not really pictures, but rather are windows to Plato's heaven...As telescopes help the unaided eye, so some diagrams are *instruments* (rather than representations) which help the unaided mind's eye" (Brown *apud* Maddy 2011, 118. My emphasis).<sup>13</sup>

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## Notes

- 1 Plato’s intellectual stimulations are *not* limited to the *explicit* requests of collaboration between writer and reader that the philosopher introduces in his dialogues. Plato is also able to elaborate intellectual stimulations whose meaning is unveiled gradually by the readers who progress rationally. I define both the explicit and the non-explicit cognitive stimuli devised by Plato in the dialogues as *epistemic games*. The nature and the features of the *epistemic games* are analyzed in my book, Saracco, S. 2017. *Plato and Intellectual Development: A New Theoretical Framework Emphasising the Higher-Order Pedagogy of the Platonic Dialogues*. Cham, Switzerland: Palgrave Macmillan. See in particular the second chapter, *The Structure of Rational Engagement in the Reading of Plato*, pp. 13-53.
- 2 Stating this I do *not* want to associate my theory with the point of view of those scholars who claim that Platonic *basic* teachings are not part of his written dialogues because they belong to his unwritten doctrines (See the Tübingen school, in particular Krämer, Hans J. 1990. Edited and translated by Catan, John R. *Plato and the Foundations of Metaphysics: A Work on the Theory of the Principles and Unwritten Doctrines of Plato with a Collection of the Fundamental Documents*. Albany: State University of New York Press and Szlezák, Thomas. 1999. *Reading Plato*. Translated by Zanker, Graham. London: Routledge). On the contrary, I do think that the *fundamental* Platonic teachings *are* in the written dialogues. The existence in this work of indications of the presence of a stage of rational evolution, complementary to the intellectual development rendered possible by the Platonic written texts, does not mean that there are fundamental concepts of Plato’s philosophy that are not part of his written words. My idea is that the basics of Plato’s thought *are* in the dialogues but the dialogues should not be considered as the final stage of cognitive evolution but as the means to reach a further stage of rational development, whose detailed description is not provided by Plato.
- 3 On this subject see Saracco, S. 2017. *Plato and Intellectual Development: A New Theoretical Framework Emphasising the Higher-Order Pedagogy of the Platonic Dialogues*. Cham, Switzerland: Palgrave Macmillan. See in particular the third chapter, *Theoretical Childhood and Theoretical Adulthood*, pp. 53-83 and Saracco, S. 2016. “Theoretical Childhood and Adulthood: Plato’s Account of Human Intellectual Development.” *Philosophia: Philosophical Quarterly of Israel*, 44 (3).
- 4 The *epistemic function* of the Forms in relation to the new theoretical framework that I have developed to explain Plato’s idea of human intellectual growth is not the subject of this piece. To know more on the topic see Saracco, S. 2017. *Plato and Intellectual Development: A New Theoretical Framework Emphasising the Higher-Order Pedagogy of the Platonic Dialogues*. Cham, Switzerland: Palgrave Macmillan. See in particular the fourth chapter, *Plato’s Forms and Scientific Modelling*, pp. 87-107.
- 5 This is not problematic: the strength of the message that I want to convey does not depend on the specific details of the reconstruction of the Platonic account of human development. A reader who thinks that the last phase of the cognitive individual growth, that I call theoretical adulthood, has to be represented using three subsections of the line segment which symbolizes intellectual development, instead of the four subsections that I have chosen to represent this phase of cognitive development, is assuming the necessity to contextualize Plato’s written words in a broader theoretical framework, represented by an extended line segment. This reader, developing this type of criticisms, is also interacting with the Platonic text, accepting the request of collaboration between writer and reader that I have emphasized as fundamental for the philosopher. This kind of criticisms does not undermine but reinforces the basics of my work.
- 6 The equal length of the subsections of my line segment does not aim at suggesting that the ancient Greek text should be revised so that the modified words would create the chance to compose unproblematically the Platonic schematization of the stages of intellectual progress using four equal subsegments (“The Revisionist Interpretation” (Foley 2008, 8-9)). I also do not want to commit myself to

the idea that “the two middle segments were not meant to be compared” (Foley 2008, 9-12). This is the way in which the length of the sectors of the line segment of the *Republic* is treated in the so-called demarcation interpretation. Its name derives from the fact that its exponents think that exists a “clear demarcation between the intended and unintended points of comparison, and such a demarcation will show that the equality of the middle subsegments can be dismissed because it falls into the latter category”(Foley 2008, 10). I am not interested here in debating whether the equality of the two middle subsegments is unintended (“The Gaffe Interpretation” (Foley 2008, 12-15)), or intended (“The Dissolution Interpretation” (Foley 2008, 15-18)). I want simply to stress the more general point that all the four subsections described in the *Republic* (R. VI 509d-511) are important for our cognitive growth but the significance of the process of human intellectual evolution cannot be fully grasped if its reconstruction is limited to these sectors.

- 7 More on this subject in the third chapter of my book (Saracco 2017), *Theoretical Childhood and Theoretical Adulthood*.
- 8 An example of how the axiomatic method works, in connection with its application to solve the first problem of Euclid's *Elements*, can be found in the third chapter of my book (Saracco 2017), *Theoretical Childhood and Theoretical Adulthood*, pp. 70-73.
- 9 Giaquinto, Marcus. 2007. *Visual Thinking in Mathematics: An Epistemological Study*. Oxford: Oxford University Press. For geometrical knowledge see in particular chapters 2-4. See also Giaquinto, Marcus. 2008. “Visualizing in Mathematics.” In *The Philosophy of Mathematical Practice*, edited by Mancosu, Paolo. New York: Oxford University Press.
- 10 On this topic see also Giaquinto Marcus. 2007. *Visual Thinking in Mathematics: An Epistemological Study*. Oxford: Oxford University Press, chapter 11, see in particular pp. 226-236.
- 11 On theoretical adulthood see the fifth chapter of my book (Saracco 2017), *Theoretical Adulthood*.
- 12 Shapiro's epistemology is efficaciously summarized in MacBride, Fraser. 2008. “Can *Ante Rem* Structuralism Solve the Access Problem?” *The Philosophical Quarterly* 58 (230), pp. 157-158.
- 13 *Acknowledgements* I would like to thank Marco Panza and Andrea Sereni for having introduced me to the subject of visual thinking and its epistemological importance. My ideas about structuralism and its relation with Plato's philosophy have been enriched thanks to Andrea Sereni's clarifications. I hope to have made the most of the time that both of them have devoted to me in the middle of the Covid-19 pandemic chaos.